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Comparatively little analysis is used in this work. The subject is confir *he mechanics of solids.* The work abounds with numerous examples, m *hich are drawn from the* examination papers of English Universities.

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Jan. 1879.

THE

PRINCIPLES

OF

ELEMENTARY MECHANICS.

BY

DE VOLSON WOOD,

PROFESSOR OF MATHEMATICS AND MECHANICS IN THE STEVENS INSTITUTE OF TECHNOLOGY.

NEW YORK:
JOHN WILEY & SONS,
15 ASTOR PLACE.
1878.

TROW'S
PRINTING AND BOOKBINDING CO.,
205-213 *East 12th St.*,
NEW YORK.

PREFACE.

THIS work is especially designed to treat of the *principles* of Rational Mechanics, and not to present a system of analysis. The analysis employed in the demonstration of principles is of an elementary character, the Calculus being entirely avoided. A few problems are solved which very properly belong to the Calculus, but the solutions have been effected by means of the well-known properties of certain curves and the principles of elementary geometry. As examples of this mode of reasoning, reference is made to the following problems: The determination of the centre of gravity of a circular arc; The time of vibration of a simple pendulum; and The quantity of flow of a liquid through a weir. A few problems are solved which involve a knowledge of Conic Sections, but these may be omitted, if desired, without detriment to the other portions of the work.

The Articles are as independent of each other as they can be practically, and at the same time present the subject in a connected manner. This feature will enable the teacher the more easily to select particular portions of the work when the whole cannot be taken.

The manner of applying the principles of the subject

is shown by means of numerous problems, examples, and exercises. The problems are of a general character and are accompanied by a full solution. The examples are numerical, and are intended to be special applications of the formulas and principles contained in the chapter of which they form a part. The exercises are a novel feature of the work. They are intended particularly to draw out and fix in the mind the general principles of the subject. The answers to the questions under this head are not always explicitly given in the text, but the principle involved in the answer is sufficiently explained there. Additional questions will doubtless suggest themselves both to the teacher and student, and in some cases conditions may be added to those given in the exercise. Thus, in the 5th Exercise, on page 23, the question may be raised whether the weight of the rope is to be considered; and if so, whether the velocity is to be uniform or variable; also, in the latter case, whether the acceleration be increasing or decreasing.

The abstract relations which exist between a force and the motion which it produces in a body, are considered early in the work. I do not consider this order as in the least essential, but I have usually presented the subject in this way to my classes, regardless of the order given in the text-book. No part of abstract mechanics possesses a greater interest than this, and the questions pertaining to momentum and energy which grow out of these relations have provoked a great deal of discussion among students in mechanics. In regard to Momentum

and Vis Viva, much of the difficulty which arises in the mind of the student in regard to them would be removed, if they are considered, as they really are, mutually independent of each other, having no common unit between them, but each having its own peculiar unit.

The demonstration of the formula for centrifugal force may appear to be unnecessarily lengthy, but I trust that the student will gain a clearer conception of the mode of action of the forces, and be better satisfied with the logic of the demonstration, by following the proof here given, than by certain of the shorter methods. The demonstration by some of the latter methods is defective, although the same result is reached by them.

The principles of energy, which plays such an important part in modern physics, have been explained, and the principles of both Kinetic and Potential Energy used in the solution of problems.

HOBOKEN, December, 1877.

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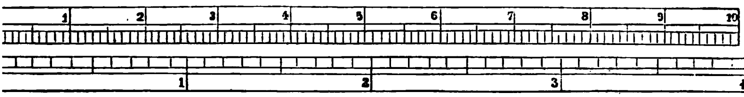
FRENCH AND ENGLISH MEASURES.

A DECIMETRE DIVIDED INTO CENTIMETRES AND MILLIMETRES.



Inches and tenths.

FRENCH MEASURES IN EQUIVALENT ENGLISH MEASURES.

MEASURES OF LENGTH.

1 Millimetre = 0·03937079 inch, or about $\frac{1}{25}$ inch.
1 Centimetre = 0·3937079 inch, or about 0·4 inch.
1 Decimetre = 3·937079 inches.
1 Metre = 39·37079 inches = 3·28 feet nearly.
1 Kilometre = 39370·79 inches.

MEASURES OF AREA.

1 sq. millimetre = 0·00155006 sq. inch.
1 sq. centimetre = 0·155006 sq. inch.
1 sq. decimetre = 15·5006 sq. inches.
1 sq. metre = 1550·06 sq. inches, or 10.764 sq. feet.

MEASURES OF VOLUME.

1 cu. centimetre = 0·610271 cu. inch.
1 cu. decimetre = 61·0271 cu. inches.
1 cu. metre = 61027·1 cu. inches.

The litre (used for liquids) is the same as the cubic decimetre.

MEASURES OF WEIGHT.

1 Milligramme = 0·015432349 grain.
1 Centigramme = 0·15432349 grain.
1 Decigramme = 1·5432349 grains.
1 Gramme = 15·432349 grains.
1 Kilogramme = 15432·349 grains, or 2.2 lbs. nearly.

TWO UNITS INVOLVED.

1 Gramme per sq. centimetre = 2·048098 lbs. per sq. foot.
1 Kilogramme per sq. metre = 0·2048098 " "
1 Kilogramme per sq. millimetre = 2·048098 " "
1 Kilogramme metre = 7·23314 foot-pounds.
= $7\frac{1}{4}$ foot-pounds nearly.

1 force de cheval = 75 kilogrammetres per second, or $542\frac{1}{2}$ foot-pounds per second nearly. 1 horse-power = 550 foot-pounds per second.

ENGLISH MEASURES IN EQUIVALENT FRENCH MEASURES

MEASURES OF LENGTH.

1 inch = 25·39954 millimetres.
1 foot = 0·304794 metre.
1 yard = 0·9143834 metre.
1 mile = 1·60932 kilometre.

MEASURES OF AREA.

1 sq. inch = 645·137 sq. milli'tres.
1 sq. foot = 0·0929 sq. metre.
1 sq. yard = 0·83609 sq. metre.
1 sq. mile = 2·59 sq. kilometres.

MEASURES OF CAPACITY.

1 pint = 0·5676 litre.
1 gallon = 4·5410 litres.
1 bushel = 36·3281 litres.

MEASURES OF VOLUME.

1 cu. inch = 16386·6 cu. milli'tres
1 cu. foot = 0·0283 cu. metre.
1 cu. yard = 0·7645 cu. metre.

MEASURES OF WEIGHT.

1 grain = 0·064799 gramme.
1 oz. avoir. = 28·3496 grammes.
1 lb. avoir. = 0·4535 kilogramme.
1 ton = 1·01605 tons.
= 1016·05 kilog.

TWO UNITS INVOLVED.

1 lb. per sq. foot = 4·88261 kilog. per sq. metre.
1 lb. per sq. inch = 0·0703 kilog. per sq. centimetre.
1 foot-pound = 0·1382 kilogrammetre.

GREEK ALPHABET.

Letters.	Names.	Letters.	Names.	Letters.	Names.
Α α	Alpha	Ι ι	Iōta	Ρ ρ	Rho
Β β	Bēta	Κ κ	Kappa	Σ σ ς	Sigma
Γ γ	Gamma	Λ λ	Lambda	Τ τ	Tau
Δ δ	Delta	Μ μ	Mu	Υ υ	Upsilon
Ε ε	Epsilon	Ν ν	Nu	Φ φ	Phi
Ζ ζ	Zēta	Ξ ξ	Xi	Χ χ	Chi
Η η	Eta	Ο ο	Omicron	Ψ ψ	Psi
Θ ϑ θ	Thēta	Π π	Pi	Ω ω	Omega

ELEMENTARY MECHANICS.

CHAPTER I.

KINEMATICS OR MOTION.

Definitions.

1. **Motion.**—As we look about us we see many objects in motion; such as men walking, birds flying, ships sailing, etc. We also know that the earth and other planets are moving through space.

2. **Rest.**—We also see many objects apparently at rest; that is, they remain in the same place in reference to surrounding objects. Thus, hills, rocks, buildings, etc., appear to be at rest.

3. **Kinematics** *is the science of motion.*—It does not consider the *cause* of the motion, but determines its measure, and the relations between different motions.

4. **Rest and motion** *are relative terms.*—A body may be at rest in reference to some objects, and in motion in reference to others. Thus, a person sitting on the deck of a ship may be at rest in reference to the objects on the ship, while he is moving with the ship over the water. But if he should run toward the stern at the same rate that the vessel is advancing, he will appear to be at rest in reference to objects on the shore, and moving in reference to the objects on deck. Objects at rest on the earth are moving through space with the earth at the rate of more than 67,000 miles per hour.

The motion of one body in reference to another also in motion is called *relative;* but in reference to a fixed object it is called *absolute* or *actual.*

5. The path *of a body is the line traced by its central point.*—If all the points of a body move in parallel lines, any one of the lines may be taken as the path. Unless otherwise stated, we will assume that the body is reduced to a mere particle. The *path* is also called the *space* over which a body moves.

Velocity.

6. Velocity *is rate of motion.*—When a body passes over equal successive portions of space in equal times, its rate is *uniform.* In all other cases it is *variable.* Thus, if a body moves uniformly from A to B in four seconds, the spaces passed over each second will be one-fourth of AB, or A–1 = 1–2 = 2–3, etc., and any one of these spaces is the *velocity.*

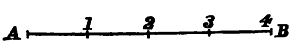


FIG. 1.

When the motion is uniform, the velocity is the space passed over in a unit of time, and the velocity is said to be constant.

If s = the space passed over uniformly,
t = the time of the movement, and
v = the velocity,

then we have, according to the definition,

$$v = \frac{s}{t}; \quad . \quad . \quad . \quad (1)$$

from which we find,

$$s = vt; \quad . \quad . \quad . \quad (2)$$

and

$$t = \frac{s}{v}. \quad . \quad . \quad . \quad (3)$$

7. **The unit of velocity** is understood to be *one foot per second*, unless otherwise stated. If other units are given, their equivalent value may be found in feet per second.

8. **Variable Velocity** *is that in which the rate of motion is constantly changing.*—The true measure in this case cannot be the space passed over during any single second, but *it is the space which would be passed over during a second if the body moved uniformly at the rate which it had at the instant considered.*

We are familiar with this fact. We say that a train of cars moved at the *rate* of, say, forty miles per hour, when it may have moved at that rate for an instant only, and in coming to rest it may have moved at all conceivable *rates* less than forty miles per hour.

9. **Geometrical Illustration.**—Variable velocity may be represented by a figure. Thus, in Fig. 2, let AB represent the time, say four seconds. Divide it into four equal parts, each of which will represent one second. At the several points of division erect ordinates, and make them proportional to the velocities corresponding to those times; the ordinate $a1$ representing the velocity at the end of one second; $b2$ at the end of two seconds, and so on. A curve $AabcC$ may be drawn through the extremities of these ordinates, such that the ordinate at any point will represent the velocity corresponding to that time. If a line ad be drawn through a parallel to AB, the number of square units in $1ad2$ will also represent the velocity; for the side 1-2 of the rectangle being unity, there will be the same number of square units in the rectangle that there are

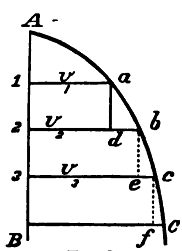


FIG. 2.

linear units in $a1$, and similarly for any other part of th figure.

The area of the figure ABC will represent the numbe of units of space passed over by the body in fou seconds.

10. The general computations *for variable velocit* belong to higher mathematics; but we are enabled t treat of some cases in an elementary manner, as will b shown hereafter.

The velocity may be found in *practical* cases with suffi cient accuracy by finding the actual space passed over b a body in a very short space of time, and considering th motion as uniform during that time.

If $\overline{\Delta t}$ (read *element of time*, or simply *delta t*) be th element of time, and

$\overline{\Delta s}$ = the corresponding space,

then, according to the definition, we have

$$v = \frac{\overline{\Delta s}}{\overline{\Delta t}}.$$

11. The path may be the arc of a circle, or any othe curved line, in which case the space will be the length o the developed line. If AB be th arc of a circle passed over in time t, we have $v = AB \div t$, as before.

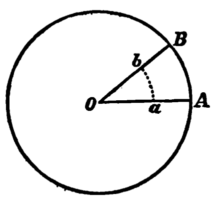


FIG. 3.

12. Angular Velocity *is the rate of angular movement. It is measured by the circular arc having unity for its radius, which would be generated by the extremity of the radius if it turned about the centre at the same rate as that of the body.* If $AB = v$, and $Oa = 1$, then ab = angular velocity.

Or, if $AB = s =$ the space passed over uniformly by the body,

$t =$ the time,

$r = OA =$ the radius of the arc AB, and

$w =$ the angular velocity,

then

$$\frac{s}{t} = \text{the velocity along } AB, \text{ and}$$

$$w = \frac{s}{t.r}.$$

13. Resolved Velocities.—Suppose that a body moves uniformly from A to B. At the extremities of the line draw two lines, AC and BC, forming a right angle at C. Then, if a body moves from A to C in the same time that one moves from A to B, the velocity of the former will equal the latter into cos. BAC.

Fig. 4.

Or, if

$v =$ the velocity along AB,

$v_1 =$ the velocity along AC, and

$\beta =$ angle BAC;

then,

$$v_1 = v \cos. \beta. \quad . \quad . \quad . \quad (1)$$

Similarly,

$$v_2 = v \sin. \beta. \quad . \quad . \quad . \quad (2)$$

when v_2 is the uniform velocity along BC.

Suppose that a body is pushed in a due southerly direction, parallel to BC, and at the same time in a westerly direction parallel to CA; if the velocities in these directions are uniform and proportional to BC and CA respectively, the resultant motion will be along a line parallel to BA, and the velocity will be proportional to BA.

The velocities represented by BC and CA are called component velocities, and that by BA is called the resultant velocity.

14. Parallelogram of Velocities.—The component velocities AC and BC may make any angle with each other. Thus, if in Fig. 5

v = the velocity along AB,
v_1 = the velocity along AC,
v_2 = the velocity along AD,
β = the angle DAC;

then, from plane trigonometry, we have

$$AB = \sqrt{AC^2 + AD^2 + 2AC.AD \cos. DAC};$$

or,

$$v = \sqrt{v_1^2 + v_2^2 + 2v_1v_2 \cos. \beta}.$$

If the angle DAC is obtuse, cos. β will be negative. If DAC is a right angle, we have cos. $90° = 0$, and

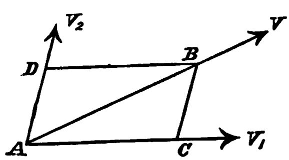


FIG. 5.

$$v = \sqrt{v_1^2 + v_2^2}.$$

This principle may be stated as follows:

If two velocities are represented in magnitude and direction by the adjacent sides of a parallelogram, the resultant velocity will be represented in magnitude and direction by that diagonal of the parallelogram which lies between these sides.

15. Triangle of Velocities.—*If two concurrent velocities be represented in magnitude and direction by two sides of a triangle taken in their order, the resultant velocity will be represented in magnitude and direction by the third side.*

Thus, in Fig. 5, if AC represent the velocity v_1 in the direction from A towards C, and CB represent the velocity v_2, acting from C towards B, then will AB represent the resultant velocity. This is evident from the preceding article.

16. Polygon of Velocities.—*If several velocities, acting successively, carry a body around a polygon, they will produce rest if they all act at the same time. And, if several velocities be represented in magnitude and direction by the successive sides of a polygon taken in their order, when they act all at the same time their resultant velocity will be represented by the closing side of the polygon.*

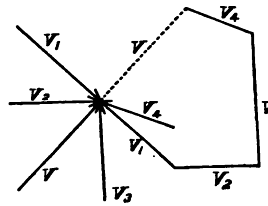


FIG. 6.

This is proved by means of the preceding article, by compounding two velocities, then their resultant and a third velocity, and so on to the last.

17. Parallelopipedon of Velocities.—*If three velocities not in one plane be represented by the three adjacent edges of a parallelopipedon, the resultant velocity will be represented by that diagonal which passes through the common point of the three edges.*

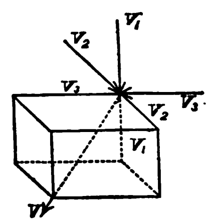


FIG. 7.

This may also be proved by the Triangle of Velocities. The resultant velocity of a body which has a southward, westward, and downward motion, will be southwesterly and downward.

18. Harmonic Motion.—If a body moves at a uniform rate around the circumference of a circle, the foot of the perpendicular from the body upon the diameter will

appear to move to and fro along the diameter with a variable velocity. Thus, in Fig. 8, if the point move uniformly around the circumference ACB, the foot of the perpendicular O will move from A towards B, thence from B towards A, and so on. The motion of the point O is said to be *harmonic*, for the *law* of its movement is similar to that of a musical string, or a tuning fork.

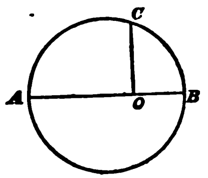


FIG. 8.

19. Periodic Motion *is that in which the motion repeats itself.*—Thus, in Fig. 8, the point that moves to and fro along the diameter AB has a periodic motion. A pendulum, as it vibrates to and fro, is another example.

20. Rotary Motion *is motion about an axis.* It is measured by its angular velocity. See Article 12.

The point about which it moves may also have a progressive velocity. Thus, the wheels of a carriage have a rotary motion about the axles, while the axles have a progressive movement. The moon revolves about the earth as a centre, and the earth not only revolves on its axis, but also revolves around the sun.

21. Problems.—1. *If the circumference of the earth at the equator is 25,000 miles, what is the velocity in feet per second of a point on the equator?*

The earth turns on its axis once in twenty-four hours, and there are 5,280 feet in a mile; hence, the velocity in feet per second is

$$v = \frac{25000 \times 5280}{24 \times 60 \times 60} = 1527 \text{ feet.}$$

2. *Required the angular velocity of the earth.*

The circumference of a circle, whose radius is unity, is $2\pi = 6.28 +$; hence, the angular velocity per hour is

$$w = \frac{6.28}{24} = 0.26 +.$$

The angular velocity per hour, in terms of degrees is

$$w = \frac{360^\circ}{24} = 15^\circ;$$

which per minute is

$$w = \frac{15^\circ}{60} = \frac{1^\circ}{4} = 15',$$

and per second is

$$w = \frac{15'}{60} = 15''.$$

EXERCISES.

1. If one body moves at the rate of 10 miles per hour, and another body, starting from the same place, moves in an opposite direction at the rate of 15 miles per hour, both moving uniformly, find the distance between them at the end of 10 minutes.
2. Which moves faster: a body moving 6 feet per second or one moving 100 yards per minute?
3. A railway train travels 90 miles in two hours; find the velocity in feet per second.
4. Two bodies start from the same place at the same time, and move uniformly at right angles with one another, one at the rate of 8 feet per second, the other at the rate of 15 feet per second; find the distance between them at the end of one second; also at the end of n seconds.
5. If a train moves uniformly at the rate of 40 miles per hour, how many seconds will it take to go 400 feet?
6. If the fly-wheel of an engine revolves 200 times per minute, what will be its angular velocity per second?
7. If a person aims to row directly across a stream at the rate of 3 miles per hour, while the stream carries him downward at the rate of 2 miles per hour, at what rate will the boat move?

8. If a wheel is rolled directly across the deck of a ship at the rate of 15 feet per second, while the ship is moving 10 miles per hour, find the velocity of the wheel in space.
9. If the velocity is one metre per hour, find the velocity in feet per second.
   (One metre is 3.28 feet nearly.)
10. Two bodies start from the same place at the same time, and move in paths which are inclined 60 degrees to each other, one moving at the rate of 5 feet per second and the other at the rate of 10 feet per second; required the distance between them at the end of two seconds.

Acceleration.

22. Acceleration *is the rate of change of the velocity.* —If the rate of motion be uniform, the velocity is constant, and there will be no acceleration. If the velocity constantly increases, the rate of change is called a *positive acceleration*, but if it constantly decreases it is called *negative.* The *rate of change* may be uniform or constantly changing. *If the acceleration is uniform, it is measured by the amount by which the velocity is increased in a unit of time;* and *if it be variable, it is measured by the amount by which the velocity would be increased in a unit of time if its rate of increase continued the same as at the instant considered.* The unit of time is usually one second. If the velocity is decreasing, the same definition applies by substituting the word *decreased* for *increased.*

23. Uniform Acceleration may be represented by a triangle. Thus, in Fig. 9, let AB represent the time—say four seconds—and BC the velocity at the end of four seconds, the body having started from rest. Divide AB into four equal parts at the points b, g, k, and draw the horizontal lines bc, ge, kh, to the line drawn from A to C;

then will bc represent the velocity at the end of the first second, ge, at the end of the second second, and so on. Draw the vertical line cd; then will de represent the increase of the velocity during the second second, and hence is the acceleration. Similarly fh will represent the acceleration during the third second, and iC that during the fourth. But $iC = fh = de = bc$; or the *velocity increases* uniformly and the *acceleration is constant*, and *the velocity at the end of the first second equals the acceleration.*

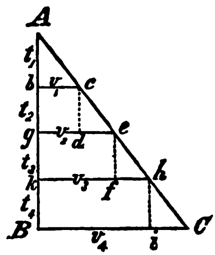


FIG. 9.

The space passed over during the second second will be represented by the trapezoid $bceg$, which is 3 times the triangle Abc. The space $gehk$ is 5 times the triangle Abc. Hence, generally, if the times are represented by the natural numbers

1, 2, 3, 4, etc., the spaces will be
1, 3, 5, 7, etc.

When the acceleration is uniform, the spaces described may be laid off on a straight line, as is shown by the line AC, Fig. 11.

24. Formulas *for uniform acceleration.*

Let

$f =$ the uniform acceleration,
$t =$ the time,
$s =$ the space passed over in the time t,
$v =$ the velocity at the end of the time t;

then, in Fig. 9, we have

$$Ab : bc :: AB : BC;$$

or

$$1 : f :: t : v;$$

$$\therefore v = ft. \quad . \quad . \quad . \quad (1)$$

Also,

$$Area\ ABC = \tfrac{1}{2} BC \times AB;$$

or

$$s = \tfrac{1}{2}vt; \quad . \quad . \quad . \quad (2)$$

and by eliminating t we have

$$s = \frac{v^2}{2f}. \quad . \quad . \quad . \quad (3)$$

Eliminating v between equations (1) and (2) gives

$$s = \tfrac{1}{2}ft^2. \quad . \quad . \quad . \quad (4)$$

When any two of the quantities f, v, t, s are given, the other two may be found from the preceding equations.

25. Initial Velocity.—The velocity which a body has at the instant t begins to be reckoned, is called *initial velocity*. This may be illustrated by Fig. 10, in which AB represents the time, DA the initial velocity, EC the final velocity, and $DECA$ the space. It thus appears that the final velocity will be that due to the acceleration *plus* the initial velocity; and the space will be that due to a uniform movement equal to the initial velocity, plus that due to the acceleration; hence, if

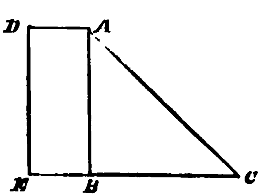


FIG. 10.

$$v_0 = \text{the initial velocity,}$$

then

$$tv_0 = ABED = \text{the space due to the ini-}$$

tial velocity; and using the same notation as in the preceding article, we have for this case

$$v = v_0 + ft. \quad . \quad . \quad . \quad (1)$$
$$s = tv_0 + \tfrac{1}{2}vt. \quad . \quad . \quad . \quad (2)$$
$$s = tv_0 + \tfrac{1}{2}ft^2. \quad . \quad . \quad (3)$$

If the acceleration is decreasing, we have

$$v = v_0 - ft, \quad . \quad . \quad . \quad (4)$$
$$s = tv_0 - \tfrac{1}{2}ft^2. \quad . \quad . \quad (5)$$

26. The Resultant of Variable and Constant Velocities.—In Fig. 11, let the horizontal velocity be constant, and the vertical motion be uniformly accelerated. If Aa is the space passed over in a horizontal direction during the first second, and Ad, the vertical space during the same time, then will the body be at the intersection of the lines drawn respectively through a and d; the former, vertical and the latter, horizontal. In a similar way its position may be found at the end of any given time.

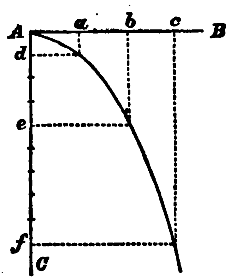


FIG. 11.

The locus of these points will be the path of the body, and will be a parabola.

In a similar way the path may be found, when both velocities are variable and acting at any angle with each other.

EXAMPLES.

1. If a body starts from rest and has a uniform acceleration of 5 feet per second, how far will it move in 10 seconds, and what velocity will it acquire?

2. A body starts from rest and acquires a velocity of 100 feet in 4 seconds; required the acceleration and the space passed over during the first second.
3. If a body starts from rest with a uniform acceleration of 32 feet per second, how far will it move during the 4th second?
4. A body has an initial velocity of 50 feet per second, and has a velocity of 100 feet at the end of 8 seconds; required the acceleration in feet per second.
5. If the acceleration is uniformly 32 feet per second; required the time necessary to pass over 200 feet, the body starting from rest.
6. If the acceleration is 20 metres in two minutes, what will be the acceleration in feet per second?
7. If the acceleration is $32\frac{1}{6}$ feet per second, what will be the value in metres per second?

CHAPTER II.

KINETICS:

(Commonly called *Dynamics.*)

27. Matter *is the substance of which bodies are composed.*—In its grosser forms we gain a knowledge of it by common experience. It is difficult, if not impossible, to define it so that a person who is not already familiar with it will gain a correct notion of it. It has certain properties, such as extension, divisibility, porosity, elasticity, etc., which it is the province of works on physics to investigate. For the purposes of Mechanical Science, it may be defined as *that upon which force acts.* But, when we consider the effect of forces upon bodies, it is necessary to know, or to assume, certain properties, especially the compressibility and elasticity of matter.

28. A Body *is a finite portion of matter.*—An *atom* is an indivisible portion of matter. It is an ideal thing, since we know nothing of its essential nature, although it has been a subject of much speculation. It is assumed that a body may be divided and subdivided, until parts might be reached which, from their constitution, cannot be again divided. A *molecule* is the smallest portion of a body which can exist without changing its nature. It is composed of two or more atoms. Thus, a molecule of water is composed of two atoms of hydrogen and one atom of oxygen, and if they be separated the result is no longer water but two distinct substances.

A *particle* is a very small body, or a small portion of a

body. It may be composed of several molecules. It has no reference to the constitution of the body. In mechanics it is considered as a material point.

According to the speculations of Sir W. Thomson, Maxwell, Tait and others, the diameter of a molecule exceeds $\frac{1}{1,250,000,000}$ of an inch, and is less than $\frac{1}{500,000,000}$ of an inch.

In order to give an idea of the minuteness of a molecule, Sir W. Thomson states that if a body of the size of an ordinary pea be expanded to the size of the earth, all the molecules expanding in the same ratio, the molecules would be between the sizes of fine shot and a cricket ball.

Force.

29. Force *is that which changes or tends to change the state of a body in reference to rest or motion.*—It is a *cause*, the essential nature of which we are ignorant of. We deal only with the *laws of its action.* These laws are determined from observation combined with certain computations. All forces do not act according to the same laws. Thus, it has been found that the *force of gravity* varies inversely as the square of the distance from the attracting body; *molecular force* varies directly as the distance between the particles; and other forces may vary according to other laws.

30. Forces are called *by different names*, according to the results produced or the phenomena presented. Thus we speak of attraction, repulsion, cohesion, friction, moving forces, accelerating forces, resisting forces, constant forces, variable forces, muscular forces, vital forces, etc.; but they are all alike in the essential quality, that they are equivalent to a *pull* or a *push*.

31. Measure of Force.—We shall assume that *the*

standard pound Avoirdupois is the measure of a unit of force, and hence, that any force is a certain number of pounds. We are familiar with the fact that forces are measured by pounds. Thus, if a spring balance of sufficient strength is placed between a locomotive and a train of cars, it will indicate the pulling force of the locomotive, whether the train remains at rest or is in motion. A push can be measured in the same manner.

Forces may also be measured by considering their effect in producing motion in a free body, as will be shown in Article 86.

32. Weight *is a measure of the attractive force of gravity upon a body.*—Weight varies directly as gravity. It has been found that the same body will weigh more in some places than others. It will weigh a trifle less on the top of a high mountain than in a deep valley, and if it could be placed where there was no attraction it would weigh nothing. It is, therefore, necessary to designate some place where a body, used as a standard, shall be weighed in order to fix the *standard pound.*

33. Standard Measures.—The standards for *weights* and *measures* are established by legal enactments. The British standard for *one pound* Avoirdupois is the weight of a certain piece of platinum kept in the Exchequer Office in London.

The British *standard yard* is the distance between two points on a certain metal rod, kept in the Tower of London, when the temperature of the whole bar is 60° F. (=15°.5C.). The *foot* is declared to be one-third of the yard.

The United States standards were copied from the British standards; but it has since been found that, on account of errors in measurement, the British yard is a trifle shorter than the American.

The French *metre* is the distance between two points on a certain bar kept for the purpose, and is nearly $\frac{1}{10,000,000}$ of the length of a meridian measured from the equator to the north pole.

The relation between these standards and certain definite quantities furnished by nature, has been determined with great care. For instance, after Capt. Kater determined very accurately the length of a pendulum which would vibrate once a second at London, compared with the standard then in use, it was declared by Parliament (5 Geo. IV.), that "the yard shall contain 36 parts of the 39.1393 parts into which that pendulum may be divided which vibrates seconds of mean time in the latitude of London, in vacuo, at the level of the sea, at temperature 62° F." After the standard was destroyed by fire, the commissioners who examined the subject reported that several reductions of the pendulum experiments were doubtful or erroneous, and, accordingly, the distance between the marks on a metallic bar was adopted as the standard, and the above ratio was discarded.

Similarly, the metre was originally declared to be $\frac{1}{10,000,000}$ of the arc of the meridian measured from the equator to the north pole, and the French government expended large sums of money in determining this distance. Certain arcs of the meridian were measured with great care, and from them the distance was computed. Afterwards it was ascertained that there had been errors in the determination, and that the distance was not the same on all meridians, for the equator is not an exact circle; but the length of the metre was not changed, and hence the declared ratio became obsolete.

Attempts have been made to establish a legal pound by declaring that a cubic foot of water at its maximum

volume shall weigh a given amount (nearly $62\frac{1}{2}$ pounds); but there is a question in regard to the temperature of the water at the maximum volume, and different experimenters do not agree as to the weight. Hence, *practically* and *actually*, the weight of a certain piece of metal remains as the standard. (See *The Metric System*, by F. A. P. Barnard, LL.D., New York, 1872.) The Avoirdupois pound is 7,000 grains. For the relation between the English and French measures see table in the Appendix. The French metre is equivalent to 39.37079 British inches, or 39.368505 American inches.

The unit of time in common use is a second of *mean solar* time. The time from any meridian passage of the sun to the following one is called a solar day, or simply a day. These days are of unequal length, caused by the variable motion of the earth along its orbit, while the time of the revolution of the earth on its axis is constant. Therefore, the *average* length of the day for an entire year is used, and called 24 hours, from which the minute and second are easily found. The *second* is the 86,400th part of a *mean solar day*.

34. A force may be represented by a straight line. Thus, in Fig. 12, the magnitude of the force may be represented by the length of the line AB, the point of application, by the point A, and its direction of action, by the arrow-head, indicating that it acts from A towards B.

$$A \longrightarrow B$$

FIG. 12.

35. The point of application of a force may be considered as at any point in its line of action. Thus, if a body is pushed with a rod, the point of application of the force may be considered as at any point along the line of the rod; and, if the rod were prolonged through, and beyond the body, the point of application may be consid-

ered as at any point on the prolonged part. The effect upon the body will be the same, whether we consider the point of application of the force at one point or another of the rod.

Inertia.

36. The Inertia *of matter means its passiveness;* that is, its inability to change its own condition in regard to rest or motion. If a body be at rest, it has no power to put itself in motion, or, if in motion, it has no power of itself to change its rate of motion. It implies that every change of motion is due to an external cause.

A body not constrained by other bodies, or in other words, perfectly free to move, is perfectly sensitive to the action of a force, so that the smallest force would move the largest body. The moon, earth, and other planets are in this condition, and each yields constantly to the action of all the others upon it. It is said, and with good reason, that every step which a man takes upon the earth changes the centre of gravity of the earth.

Inertia is not a force. But on account of the perfect passiveness of matter, if a force act upon a free body, the *effect* of the force will be stored in the body; and the body, in being brought to rest by a resistance, will produce the same *effect* as that which was stored in it.

The exact relations which exist between the motion of bodies and the force which produces it, were established only after a long series of observations, experiments and deductions. It is difficult to prove them by direct experiments, for it is difficult to realize the condition of a perfectly free body. Every body is liable to be resisted by friction, or the air, or other medium; and if the body be placed in a vacuum, the range of its motion will be limited

If a body be thrown into the air, it not only meets with the resistance of the air, but will be constantly under the action of the force of gravity. Delicate experiments have, however, confirmed all the fundamental principles of motion. Their truth is also shown from the fact that all problems pertaining to the motion of bodies, not only on the earth but in the solar system, solved in accordance with these principles, give results which agree with the results of observation. The times and places of eclipses are predicted, and the positions of the planets are foretold, by means of formulas which grow out of these fundamental principles. No truths in science have become more firmly established.

Molecular Motions—Definitions.

37. Molecular Motions.—We have thus far spoken of the motion of bodies, but a close examination shows that the *particles* of a body may have a motion in reference to each other. Thus, when a tuning-fork or a bell is struck the particles are put into a rapid **vibratory motion**, producing sound, which is transmitted to the ear by the vibrations of the air. Heat expands bodies, an effect which must be caused by the separation of the particles of the body. It is believed that the molecules or atoms of every solid are made to vibrate when it is struck.

38. Mechanics *is the science which treats of the action of forces.*—It investigates the *laws* which govern the action of forces; the conditions of the equilibrium of bodies; the motion of bodies under the action of forces; the flow of liquids and gases; and the movement of the particles of bodies.

39. Kinetics *treats of the movement of bodies under the action of forces.*—This branch of the subject has

usually been called *Dynamics*, a term which more properly pertains to the abstract *doctrine of forces*, and hence, as such, would include a portion of both Kinetics and Statics.

40. Statics *treats of the conditions of the equilibrium of solids.*

41. Molar *mechanics* treats of the action of forces upon solids.

42. Molecular *mechanics* treats of the movement of the particles of a body.

43. Hydrostatics *treats of the equilibrium of liquids.*

44. Hydrodynamics *treats of the movement of liquids.*

45. Pneumatics treats of the laws of pressure and movement of air and other gaseous bodies.

46. Thermodynamics *treats of the mechanical properties of heat.*

47. Rotation of Bodies.—A force acting upon a body may produce both translation and rotation at the same time. Thus, a boy strikes a stick with a club, and if he does not strike it directly opposite the centre it will rotate at the same time that it moves forward. The solution of problems involving the rotation of bodies are generally more difficult than those involving translation only. If all the points of a body move in parallel straight lines, the motion is that of simple translation, and will be substantially the same as if we consider the body reduced to a mere particle. Hence, in discussing the subject of translation it is common to speak of the body as a particle.

EXERCISES.

1. If a body be suspended by a very long, fine string, how much force will be required to move it sideways?
2. If a ship could float on water without any resistance from the water or air, how much force would be required to move it? If it were

moving, how much force would be required to deflect it from its path?

3. Why does not every body move when acted upon by force?
4. If four spring balances are connected end to end, and a man pulls with a force of 100 pounds at one end of the four, what will be the force exerted at the other end, and what will each balance indicate?
5. If a constant pull of 500 pounds is exerted at one end of a rope, which is attached to a boat at the other end, will the force exerted at the other end be 500 pounds when the boat is in motion?
6. Suppose that two heavy sleds are placed on perfectly smooth ice and connected by a flexible cord; if a boy draws them by pulling with a *constant* force of 10 pounds on one of them by means of a rope or otherwise, so as to pull both sleds in the same direction, will there be a force of 10 pounds exerted on the other one? That is, will the tension on the connecting cord be 10 pounds?

   (*Perfectly smooth* is intended to imply that there is no resistance between the sleds and ice.)
7. If, in the preceding exercise, the boy ceases to pull, but all the other conditions remain the same, what will be the tension upon the connecting cord?

Newton's Three Laws of Motion.

48. Sir Isaac Newton expressed the fundamental principles of the relations of force to the movement of bodies in the form of three laws or axioms. These *principles* have already been given in the preceding articles, but these *laws* are referred to so frequently, and express so clearly and correctly the fundamental principles of the motion of bodies, that we cannot do better than present them in this place.

49. First Law.—*Every body continues in a state of rest or of uniform motion in a straight line, unless acted upon by a force which compels a change.*

50. Second Law.—*Change of motion is in proportion to the acting force, and takes place in the direction of the straight line in which the force acts.*

Observe that the *resultant motion* is not necessarily in

[illegible] direction.

51. **Third Law.**—*To every action there is an equal and opposite reaction.*

That is, when two bodies press one against the other, the action from one is exactly equal but opposite to that from the other. If one presses a stone with his finger, the finger is pressed equally by the stone.

The term *motion* in these laws means more than velocity; it really means *momentum*, a term which will be defined hereafter.

Combined Forces.

52. **Parallelogram of Forces.**—If a force F, acting singly, would cause a particle at A to describe the line [illegible] in a time t, and a force [illegible] also acting singly, would cause the particle to pass [illegible] AD in the same time; [illegible] if they act jointly on [illegible] same particle, they will [illegible] it to describe the line [illegible] the diagonal of a para[illegible]

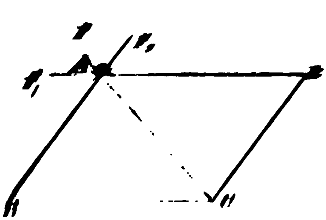


FIG. 19.

[illegible], constructed on AB and AD, in the same [illegible]

This follows directly from the Second Law. For, the force F_2 will cause the particle to reach a line DC, which is parallel to AB, in the same time, whether F_1 acts or not; and similarly F_1 will cause it to reach a line BC, parallel to AD, in the same time; and hence, when they act jointly, it will be found at the intersection of DC and BC, or at C, at the end of the time; and, as it must move in a straight line, it will move along the diagonal AC.

This is nearly the same as Article 14, only here we consider the cause of the motion, while there the velocities only were considered.

53. Resolved Forces.—If a force F would cause a particle to describe the line AC uniformly in a given time, it may be resolved into two forces, F_1 and F_2, one of which will cause it in the same time to describe the side AB, and the other, AD of a parallelogram, of which AC is the diagonal. This is the converse of the preceding article. The result, however, is indeterminate, since AC may be the diagonal of an indefinite number of parallelograms.

54. Components.—The forces F_1 and F_2 are called *component forces;* and F is the resultant.

55. The Resultant of two or more forces which act upon a single particle, is a force which will produce the same effect as all the other forces combined.

56. Relation between two Forces and their Resultant.

Let

$$F = AC; \quad F_1 = AB = DC; \quad F_2 = AD;$$
$$\alpha = CAD; \quad \beta = CAB = ACD;$$
$$\theta = DAB = 180^\circ - ADC.$$

The triangle ADC gives

$$\frac{AD}{\sin ACD} = \frac{DC}{\sin DAC} = \frac{AC}{\sin ADC};$$

or,

$$\frac{F_2}{\sin \beta} = \frac{F_1}{\sin a} = \frac{F}{\sin \theta}.$$

57. Rectangular Components.—Let the components F_1 and F_2 make a right angle with one another; and a and β be the same as in the preceding article. Then the triangle ADC gives

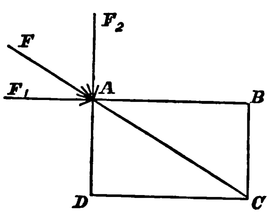


FIG. 14.

$$F_1 = F \cos \beta;$$
$$F_2 = F \cos a = F \sin \beta.$$

Squaring and adding, observing that $\sin^2 a + \cos^2 a = 1$, we have

$$F^2 = F_1^2 + F_2^2.$$

58. If several forces act along the same line upon a body, whether in the same or opposite directions, their resultant is the algebraic sum of the several forces.

Let $R =$ the resultant, then

$$R = \Sigma F.$$

EXERCISES.

1. When a body is thrown horizontally into the air, why does it fall towards the earth? (See the SECOND LAW.)
2. The planets are free bodies in space; how much force does it require to deflect them from their course?
3. A stone, whose weight is 500 pounds, rests upon another stone whose weight is 2,000 pounds; what is the reaction of the latter against the former?

4. If two boats of equal size, offering equal resistances to movement on water, are connected by a rope, and a man in one of the boats pulls on the rope, drawing the boats toward one another, at what point between them will they meet? If a man in the other boat also draws in the rope, at what point will they meet?
5. If a person is on ice, which moves in a due easterly direction at the rate of 1 mile per hour, and he walks on it at the rate of 3 miles per hour, in what direction must he travel so that his resultant course shall be due south?
6. If a ship is sailing south-east at the rate of 8 miles per hour, and the tide is carrying it due east at the rate of 3 miles per hour, what is its actual course, and its velocity?
7. A ship is sailing due south-east at the rate of 10 miles per hour; what is its velocity in a southerly direction?
8. If a ball is placed on the floor of a railroad car, and is perfectly free to roll, will it change its position if the velocity of the train is gradually increased? Why would it rush toward the forward end in case of a collision with another train?
9. The force F_1 equals 20 pounds, F_2 equals 30 pounds, and the angle between them is 60 degrees; required the value of their resultant and the angle between it and the forces F_1 and F_2 respectively.

Constant Force.

59. A constant force *is one which acts with a constant intensity.*

An *incessant force* is one which acts constantly, but with a *variable* intensity.

60. *A constant force applied to a free body, and acting along the line of motion, produces a constant acceleration.*

For, according to the FIRST LAW, if no force be applied the velocity will remain constant, and, according to the *second law*, the change of velocity will be proportional to the force; and as the force remains constant the change of velocity must be the same for each unit of time, that is, *constant.*

If the force be applied so as to produce an increase of

velocity, the acceleration will be *positive*; but if it causes a decrease of velocity, the acceleration will be *negative*. In the former case the velocity would increase indefinitely, but in the latter case the body might be brought to a state of rest. For instance, a body thrown horizontally on ice, or a train of cars, moving after the locomotive is detached, would be brought to rest by friction. An active force, operating to destroy the velocity, may, by its continued action, produce a velocity in an opposite direction. Thus, when a train of cars is in motion, the locomotive may be reversed and push against the train to stop it, and by continuing its action may finally produce a velocity in an opposite direction.

61. A constant moving force is one in which the resultant of all the forces is constant. Thus, when a locomotive draws a train of cars, it may exert a constant *moving* force for a time; that is, there will be a constant force for producing motion; but, as the velocity increases, the resistance of the air increases, and usually, after a short time, the whole power of the locomotive is exerted in overcoming the friction and resistance of the air, and produces no increase of velocity. In the latter case, although the locomotive may exert a constant force, it is not called a constant *moving* force, for the pulling force of the locomotive is exactly neutralized by the resistances of the train.

62. The movement of a body under the action of a constant moving force is illustrated by Fig. 9, because the force produces a constant acceleration. Hence, the formulas of Article 24 give the relations between the time, space and velocity, when the acceleration is known. The line of action of the force is supposed to be in the direction of motion.

63. Normal action.—If a force acts constantly normal to the path described by the body, it will not affect the velocity. Thus, if a body be connected to a point by a string, so that the body will describe the circumference of a circle, the tension of the string will not change the velocity. The force of the string and the velocity of the body may both remain *constant.*

64. The force of gravity is one of the forces of nature. It is an attractive force, tending to draw bodies towards each other. It always manifests itself wherever there is matter. It is the force which gives weight to bodies, and causes unsupported bodies to fall to the earth. It holds the planets in their orbits. It acts through bodies without being diminished in its intensity, and upon the most central portions of a body with the same intensity as if the external portions were removed.

65. The Law of Universal Gravitation is as follows: *Every particle* ATTRACTS *every other particle in the* DIRECT *ratio of its mass, and in the* INVERSE *ratio of the square of its distance.*

This law was discovered by Sir Isaac Newton in 1666, but, on account of an erroneous value of the diameter of the earth which was then used, he was not able to prove it at that time. But in 1682 it was found, from new measurements of an arc of the meridian, that the correct diameter was about $\frac{1}{17}$ greater than the value which he had previously used; and with this value he fully demonstrated the law.

Let $M =$ the mass of a body A;
$m =$ the mass of a body B;
$D =$ the distance of the body A from a body C;
$d =$ the distance of the body B from a body C;

then

$$\frac{\textit{attraction of } A \textit{ upon } C}{\textit{attraction of } B \textit{ upon } C} \propto \frac{\frac{M}{D^2}}{\frac{m}{d^2}} \propto \frac{d^2}{m} \cdot \frac{M}{D^2}.$$

66. *The force of gravity at any place on the earth remains constant.*—This is shown by the fact that the weight of a body is always the same at the same place; also, that a body always falls the same distance in a vacuum in the same time at the same place.

It is well known, however, that upheavals are taking place in some parts of the earth and depressions in others, and these doubtless produce exceedingly slight changes in the weight of a body, but no apparatus at the present day is sufficiently delicate to measure these changes, if they actually exist; hence, we may say that the force of gravity is at least *practically* constant at every place.

67. *Determination of the* **acceleration produced by gravity** *on a body falling freely in a vacuum.*—We speak of a *vacuum* because it is shown experimentally that all bodies fall the same distance in the same time in a vacuum; and also because it is necessary to exclude the resistance of the air in determining the full effect of gravity. This is a problem the *exact* solution of which involves considerable knowledge of mechanical principles and great skill in making observations. These *principles*, so far as they involve the use of the pendulum, will be explained hereafter, but in this place we can only describe the *process*.

By means of Atwood's machine an approximate value of the acceleration may be determined.

By means of delicate machinery, and a refined system of making observations, the space and time may be observed directly.

But the most reliable method, or, at least, that most commonly used, is by means of a pendulum. Any body vibrating on an axis, under the action of the force of gravity, is called a *pendulum.* If the vibrating body has perceptible size, it is called a *compound pendulum.* A *simple pendulum is a material particle suspended on a line without weight,* and hence, is an *ideal* pendulum, but it has a *real mathematical* signification. A simple pendulum may always be found which will vibrate in the same time as a compound one.

If $l =$ the length of a simple pendulum;
$t =$ the time of one vibration in seconds;
$g =$ the acceleration due to gravity in a vacuum;
$\pi = 3.141592 =$ the ratio of the diameter of a circle to its circumference;

then

$$t = \pi\sqrt{\frac{l}{g}};$$

from which we find

$$g = \frac{\pi^2 l}{t^2}.$$

To determine t, the number of oscillations for a given time, say 10 or 20 minutes, is observed, and this number, divided by the number of seconds in the time, gives t. The length l is measured directly. These quantities, substituted in the preceding formula, will give the value of g, which in this latitude is about $32\frac{1}{6}$ feet. In this way the intensity of gravity has been found at different places on the earth's surface.

68. The intensity of gravity varies with the latitude.—By means of the experiments indicated in the preceding article, it has been found that the force of

gravity is least at the equator, and increases with the latitude both north and south of the equator. The value of g for any latitude L may be found *approximately* by the formula

$$g = 32.1726 - 0.08238 \cos 2L;$$

but no formula will give the *exact* relation between g and L. At the equator $L = 0$, and at the poles $L = 90°$; hence we have

at the equator $g_0 = 32.0902$ feet, and
at the poles $g_{90} = 32.2549$ feet.

69. Weight variable.—According to the preceding article, a body will weigh less on the equator than at any other place on the surface of the earth. But this difference could not be detected by a common beam balance, for, a diminution of the weight at one end of the beam would be exactly the same as that at the other, and if the bodies balanced at one place on the surface of the earth they would balance at every other place. The difference, however, might be detected by a spring balance, Fig. 15, for the more the body weighed the more it would pull the index down.

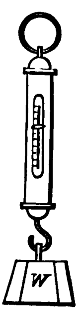


FIG. 15.

If a body be elevated one mile above the surface of the earth, it will, according to Article 65, lose about $\frac{1}{2000}$ of its weight. These variations being small, we may, for the purposes of this work, consider g as constant, and equal to $32\frac{1}{6}$ feet.

70. *There are* **three causes for the variation of gravity** *on the surface of the earth.*—*First*, the earth is an oblate spheroid, the axis of which coincides with the axis of the earth; and those bodies on the equator, being more remote from the centre of the earth than those at

higher latitudes * will be attracted with less intensity; *second*, the revolution of the earth on its axis produces a so-called "centrifugal force" (to be explained hereafter), which tends to throw bodies from the surface, thus diminishing their weight; and *third*, the heterogeneous character of the substance of the earth.

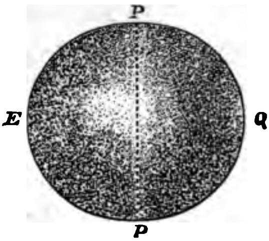


FIG. 16.

The form of the earth is supposed to be due to the attraction of the particles for each other and to the centrifugal force caused by the rotation on its axis while the substance of the earth was in a plastic state.

71. The atmosphere resists *the movement of bodies in it;* and hence, the velocity of bodies under the action of any force is less than it would be in a vacuum. The attraction of the earth being the same on each particle of a body, a light body would fall as rapidly as a heavy one if there were no resistances to their movements; and this is confirmed by experiment, by letting bodies fall in a vacuum. The resistance of the air varies with the surface against which it acts, but in falling bodies the ability to overcome this resistance varies as the weight of the body; hence, heavy bodies fall faster than light ones in the air. But the velocities of heavy bodies, such as iron, stone, brass, etc., falling 100 to 200 feet, do not differ much from

* As determined by

	The semi-polar axis is	and the equatorial radius is
Bessel	20,853,662 feet;	20,923,596 feet.
Airy	20,853,810 feet;	20,923,713 feet.
Clarke	20,853,429 feet;	20,923,161 feet.

The equatorial diameter of the earth is about 26 miles longer than its axis.

each other; and for compact masses of such mat falling in air we use $32\frac{1}{6}$ feet for g.

Formulas for Falling Bodies.

72. Bodies falling from rest.—The acceleration l constant, we have only to substitute its value in the tions of Article 24. Making $f = g$ and $s = h$ in formulas, we have,

$$v = gt; \quad . \quad . \quad . \quad . \quad .$$

$$h = \tfrac{1}{2}gt^2; \quad . \quad . \quad . \quad .$$

$$\therefore h = \tfrac{1}{2}\,vt = \frac{v^2}{2g}; \quad . \quad . \quad .$$

$$\text{and } t = \frac{v}{g} = \left(\frac{2h}{g}\right)^{\frac{1}{2}} = \frac{2h}{v}. \quad . \quad .$$

73. If a body is projected downward *with a vel* v_0, we make $f = g$ and $s = h$ in Article 25, and have

$$v = gt + v_0. \quad . \quad . \quad .$$

$$h = \tfrac{1}{2}gt^2 + v_0t. \quad . \quad . \quad .$$

74. If a body is projected upward *with a vel* the acceleration becomes negative, and the equatio the preceding article become

$$v = v_0 - gt. \quad . \quad . \quad .$$

$$h = v_0t - \tfrac{1}{2}gt^2. \quad . \quad . \quad .$$

75. Problems.—1. *If a body is projected upward a velocity of* 100 *feet per second, required its heig the end of* 2 *seconds.*

Equation (8) gives

$$h = 100 \times 2 - \tfrac{1}{2} \times 32\tfrac{1}{6} \times 4 = 135\tfrac{2}{3} \text{ feet.}$$

2. *In the preceding problem what will be the heig the end of* 8 *seconds?*

We have

$$h = 8 \times 100 - \tfrac{1}{2} \times 32\tfrac{1}{6} \times 8^2 = -229\tfrac{1}{3} \text{ feet};$$

that is, at the end of 8 seconds, the body will be $229\frac{1}{3}$ feet *below* the starting point.

3. *In Problem* 1, *what will be the greatest height of ascent?*

When it is at the greatest height v will be zero in equation (7); hence

$$v_0 = gt.$$

$$\therefore t = \frac{v_0}{g} = 3.1 + \text{sec.}$$

and this in equation (8) gives

$$h = 100 \times 3.1 - \tfrac{1}{2} \times 32\tfrac{1}{6} \times (3.1)^2 = 156.4 \text{ feet.}$$

4. *If a body is projected upward with a velocity of* 200 *feet per second, required its height when the velocity is* 100 *feet per second.*

From equation (7), we find for the time

$$t = \frac{200 - 100}{32\frac{1}{6}} = \frac{600}{193} \text{ seconds,}$$

which substituted in equation (8) gives

$$h = 200 \times \frac{600}{193} - \tfrac{1}{2} \times \frac{193}{6} \times \left(\frac{600}{193}\right)^2 = 466.3 \text{ feet.}$$

EXAMPLES.

1. A body falls from rest through a height of 100 feet; required its final velocity. (Let $g = 32\frac{1}{6}$ feet.)
2. A body falls from rest and acquires a velocity of 300 feet; required the time.
3. A body is projected upward with a velocity of 100 feet per second; what will be the greatest height of ascent?

4. If $g = 32\frac{1}{6}$ feet per second, what will be the acceleration per minute.
5. A metre is 3.28 feet (nearly); if the unit of time were 2 seconds what would be the acceleration?
6. If a body is projected downward with a velocity of 25 feet per second, what will be the velocity after it has fallen 120 feet?
7. In the preceding example, what will be the time of descending 150 feet?
8. At the instant a body is dropped from a point A, another body is projected upward from a point B, vertically under A, and they meet at the middle of AB; required the velocity of projection from B.

   Ans. $(AB \times g)^{\frac{1}{2}}$.
9. A body is let fall into a well, and 4 seconds afterward it is heard to strike the water; if the velocity of sound is 1130 feet per second, required the depth of the well.

   Ans. 231 *feet.*
10. A body is projected downward from a point A with a velocity of v feet per second, and another body is projected upward from a point a feet below the former, with a velocity of V feet per second; required their point of meeting.

EXERCISES.

1. If a boy draws a load on a sled with an increasing velocity, will he exert any more force than if he draws it at a uniform rate?
2. Why will a body fall more rapidly at the foot of a mountain than at its top?
3. Why may some light bodies fall more rapidly at the top of a mountain than at its foot?
4. If a body, whose weight is 5 lbs., attracts a particle at a distance of 5 feet with a force of $\frac{1}{100}$ of an ounce, with what force will

another body, whose weight is 25 lbs., attract the same particle at a distance of 15 feet?

5. Which will vibrate in a shorter time, a pendulum 10 inches long or one 15 inches long?
6. If a pendulum vibrates once each second at New York, will the time of a vibration of the same pendulum be more or less than a second at the equator?
7. If a merchant weighs iron in New York and sends it to some port near the equator; will it gain or lose in weight if it be weighed in both places with the same *beam* scales? Will it gain or lose if weighed with the same *spring* balances?
8. What is the value of g ($32\frac{1}{6}$ feet) in metres per second?

Attraction of Homogeneous Shells.

76. Problem.—*The attraction of a perfectly homogeneous, spherical shell is the same upon a particle placed anywhere within it.*

Let $ABCDE$ be a section of an indefinitely thin spherical shell, and O any point within it. Draw lines BOD and AOC through the point O, making an indefinitely small angle with each other, and consider AOB and COD as two cones having their vertices at O, and their bases AB and CD in the surface of the sphere. The quantities of matter in each will be directly as their bases, but at the limit the triangles are similar, and the bases will be as AO^2 to OC^2. The attraction of each varies as the quantity of matter and inversely as the square of the distance from O; (see Article 65) that is,

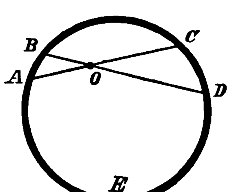


FIG. 17.

$$\frac{\textit{Attraction of } AB}{\textit{Attraction of } CD} \propto \frac{(AO)^2 \times \frac{1}{(AO)^2} = 1}{(OC)^2 \times \frac{1}{(OC)^2} = 1};$$

hence, the attraction of the bases will equal one another, and, being in opposite directions, will neutralize one another's effect. Similarly, conceive that the shell constitutes the bases of an indefinitely large number of cones, then, according to the above reasoning, the attraction of all the matter on one side of any straight line drawn through O will exactly neutralize that on the other side. The same may be proved for any other point.

77. Problem.—*If the earth were a homogeneous, hollow sphere of uniform thickness, a body placed at any point within the hollow would remain at rest.*

For, according to the preceding article, the attraction of any of the spherical shells of infinitesimal thickness upon any point within it will be zero, and hence, the effect of all of them upon the same point will be neutralized.

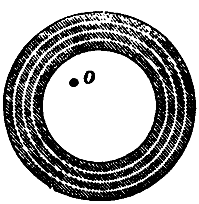


FIG. 18.

It follows from this that if the earth were solid and composed of homogeneous, concentric shells, though they varied according to any law from the centre to the surface, the resultant attraction upon a particle at the centre of the earth would be zero.

78. Problem.—*If the earth were a homogeneous solid sphere, the resultant attraction upon any point within it would vary directly as its distance from the centre of the earth.*

Suppose that a particle is at a distance x from the centre; then, according to the preceding problem, the resultant attraction of the shell outside of x will be zero; and that portion of the sphere whose radius is x will attract directly as its quantity of matter and inversely as the square of the distance of the particle from the centre of the sphere.

The quantity of matter will be directly as the volume, or as

$$\tfrac{4}{3}\,\pi\,x^3;$$

and hence, the attraction will vary as

$$\tfrac{4}{3}\,\pi\,x^3 \div x^2, \text{ or as } \tfrac{4}{3}\,\pi\,x;$$

that is, directly as the distance of the particle from the centre.

Weight—Mass—Density.

79. Weight.—The weight of a body has already been defined, in Article 32, *as a measure of the attractive force of the earth upon the body.*

Weight is not essential to matter. According to Article 76, if a body were placed at the centre of the earth it would weigh nothing. Similarly, if placed at a certain point between the earth and moon it would lose its weight. According to the preceding articles its weight depends upon its position; or more specifically, upon the attraction of the earth upon it.

80. Mass *is a term used to express quantity of matter.*—This, as we have seen, is independent of the weight of the body; in other words, it is constant for the same body. The ratio of the weights of two bodies in *vacuo*, determined at the same time, whether on the equator, the top of a mountain, within the earth, or at any place in the universe, is constant. Hence, if two bodies, one weighing 2 pounds and the other 10 pounds, be balanced on a lever at any place, they would remain balanced if taken to a place where the bodies weighed 1 and 5 pounds respectively, or 4 and 20 pounds respectively.

81. Measure of Mass.—*The mass of a body equals its*

weight at any place divided by the acceleration due to gravity at the same place.

It must be measured in such a way as to give the same value wherever determined. As already shown, the weight varies directly as the force of gravity, and the acceleration due to gravity also varies as the same force; hence, the ratio of the weight to the acceleration will be constant for all places, both being determined at the same place.

If

$M =$ the mass of a body;
$W =$ the weight of the body at any place, and
$g =$ the acceleration due to the force of gravity at the same place;

then,

$$M \propto \frac{W}{g},$$

which may be put in the form of an equation by introducing a constant c, hence,

$$M = c\,\frac{W}{g}.$$

Since c is arbitrary, the unit of mass may be so chosen that c will be unity, in which case we have

$$M = \frac{W}{g};$$

which is the value used in Mechanics.

The weight in this case must be determined by a spring balance, or its equivalent, which must weigh correctly a standard pound, or multiples thereof.

For ordinary practical purposes it is only necessary to divide the weight of a body, determined at any place on the earth with a pair of good scales, by $32\frac{1}{6}$.

The mass of a body might also be found by weighing it at any place with a pair of beam scales, using as weights *standard units of mass;* that is, $32\frac{1}{6}$ standard pounds of weight.

82. *The mass of a body is the number of pounds that a body would weigh at that place where the acceleration due to gravity is one foot per second, the weight being determined by a standard spring balance.*

In the last equation of Article 81 make $g = 1$, and letting W_1 be the corresponding weight, we have

$$M = W_1.$$

One such place is in the earth, at about $\frac{1}{32}$ of the radius of the earth from the centre, or less than 125 miles from the centre of the earth. Another point is about 22,000 miles from the centre; for if x be the distance, and the radius of the earth be called 4,000 miles, we have, according to Article 65,

$$(4000)^2 : x^2 :: 1 : 32\tfrac{1}{6};$$

$$\therefore x = 22{,}686 \text{ miles}.$$

Since the weight of the same body varies directly as the force of gravity, we have

$$M \, \textit{pounds} : W \, \textit{pounds} :: 1 \, \textit{foot} : g \, \textit{feet};$$

$$\therefore M = \frac{W}{g}.$$

In this expression g may be considered as an abstract number, being the *ratio* of the acceleration due to gravity at two different places.

83. The Unit of Mass *is a body which weighs* $32\frac{1}{6}$ *standard pounds.*

For, if $M = 1$ in the preceding article, we have

$$W \, \textit{pounds} = g \times 1 \, \textit{pound} = 32\tfrac{1}{6} \, \textit{pounds}.$$

The term *pounds* is used in a double sense. We u *pounds of weight* and *pounds of mass*, but no ambigui arises on account of it, for the language of the proble will always determine which is referred to.

84. Analytical expression for Weight.—From th last equation of Article 82 we have

$$W = Mg\,;$$

an expression which is useful in the solution of ma problems.

85. Density relates to the compactness of matter. the mass of a body be uniform throughout the volume, *t density is the mass of a unit of volume.*

Let

$M =$ the mass of a homogeneous body,
$V =$ the volume of the body,
$D =$ its density;

then,

$$D = \frac{M}{V}. \quad \therefore M = DV.$$

If $V = 1$, then

$$D = M.$$

Substitute the value of M from the last equation Article 82, and we have

$$D = \frac{W}{gV}.$$

If the density be variable the density at any point of body will equal the mass of a unit of volume having t same density throughout the unit as that at the point the body considered.

The unit of volume may be taken as a cubic inch, fo or any other standard measure.

EXERCISES.

If the earth were a homogeneous sphere, and a body on its surface weighed 10 lbs., what would be its weight if placed at a point half way between the surface and centre?

If a body weighs 10 lbs. on the surface of the earth, what would it weigh if elevated to a point above the surface equal to the radius of the earth?

If the earth were a homogeneous spherical shell of finite thickness, what would be the weight of a body placed anywhere within the hollow?

(The earth is referred to in these questions because the conditions could not be realized by a hollow sphere on the surface; for gravity would exert its full force on a body placed in the hollow of such a sphere.)

In the preceding example, if a body weighed 10 lbs. on the surface of the earth, at what place in the shell must it be placed that its weight shall be 5 lbs.?

If the earth were a homogeneous shell, and a body were dropped from the surface into the hollow, would the motion be accelerated or not as it passes through the shell? And as it passed across the hollow would its motion be accelerated or not, no allowance being made for the resistance of the air?

If a person were placed in the hollow described in the preceding question, and should jump from one side toward the centre of the sphere, where would he stop? Could he stop at the centre if he desired to?

If a person were placed at one extremity of a diameter of the hollow referred to in exercise 5, and a ball of equal mass placed at the other extremity, and the person should pull on the body by means of a string, where would they meet? If he pulls for an instant and then ceases to pull, will they meet? If he pulls with a constant force until they meet, will their acceleration be uniform or variable?

In the preceding question, if the ball has half the mass of the person, which will move faster, the ball or the person?

If a person were placed at the centre of the hollow sphere of exercise 5, and not able to reach anything, could he move away from the centre by his own exertions? If he had a ball and should throw it away, would the person move away from the centre?

(He could not *drop* the ball, for it would not move in any direction unless *started.*) What would be their relative directions of motion? Would they move in straight or curved lines?

10. If a body weighs 100 standard pounds, how many pounds of mass does it contain?
11. If a body whose volume is 2 cubic feet weighs 200 lbs., what is its density?
12. If a body weighs 5 kilogrammes, what is the mass in pounds?
13. If the weight of a body whose volume is 1 cubic metre is 3 kilogrammes, what is the density in pounds per cubic foot?
14. If a hole were made through the centre of the earth from surface to surface, and a ball were dropped into it, would it stop at the centre? Where would it stop if the hole were a vacuum? At what point would it move with the greatest velocity?

Dynamic Measure of Force.

86. Value of a Moving Force.—Forces are compared by their effects. If a force F, acting as a constant pull or push on a body perfectly free to move in the direction of action of the force, produces an acceleration f, and gravity acting on the same body with a constant force W, equal to the weight of the body, produces an acceleration g, we have

$$F : W :: f : g$$

$$\therefore F = W\frac{f}{g} = Mf;$$

that is, *a constant moving force of F pounds equals the product of the mass into the acceleration in feet per second.*

From the last equation we have

$$f = \frac{F}{M} = \frac{F}{W}g.$$

87. Effective Force.—Only a portion of the forces which act upon a body, may be *effective* in moving it, the

others neutralizing each other. Thus, when a locomotive draws a train of cars, a portion of the pulling force is directly neutralized by the resistance of the air, friction on the track, and other resistances of the train. If the pulling force exceeds the resistances, the *excess will be the effective pulling force.* When the resistances equal the pulling force the motion will be uniform.

88. Remark.—To find the space passed over by a body, and the velocity produced in a given time by the action of a constant effective force, find the value of f from Article 86, and substitute its value in the equations of Article 24.

89. Problems.—1. *If a piston be driven a portion of the length of a cylinder by a constant steam pressure, at what point must the pressure be instantly reversed so that the full stroke shall just equal the length of the cylinder, the cylinder being horizontal, and the piston moving without friction?*

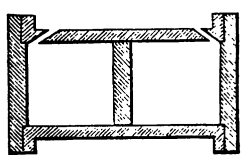


FIG. 19.

At the middle of the stroke. Whatever velocity is generated by the action through one half of the stroke will become neutralized by the same pressure acting in the opposite direction through the remaining half.

2. *In the preceding example, what will be the velocity at the centre of the cylinder?*

Let $F =$ the total pressure of the steam,
$W =$ the weight of the piston,
$s =$ one-half the length of the cylinder;

then, from Article 86, we have

$$f = \frac{F}{W} g;$$

and, if the piston starts at one end, we substitute this value in equation (3) of Article 24, and find

$$v = \left(\frac{2Fgs}{W}\right)^{\frac{1}{2}}.$$

3. *A body whose weight is W is moved horizontally on a frictionless surface by the pull of a constant force F; required the space passed over in a time t, and the velocity acquired.*

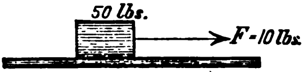


FIG. 20.

Here we have,

$$f = \frac{F}{W} g;$$

hence, from Article 24, we find

$$s = \frac{Fg}{2W} t^2;$$

and

$$v = \left(\frac{2Fgs}{W}\right)^{\frac{1}{2}}.$$

4. *If a body, whose weight is W on the surface of the earth, be placed in the hollow sphere described in Article 76; required the constant pull or push necessary to produce a velocity v in time t.*

From Article 86 we have

$$f = \frac{F}{W} g;$$

which, substituted in equation (1) of Article 24, gives

$$v = \frac{F}{W} gt;$$

from which we find

$$F = \frac{Wv}{gt}.$$

EXAMPLES.

If a piston weighs 100 lbs., and the constant steam pressure is 2,000 lbs., what will be the velocity acquired in moving over 12 inches?

In the third problem above, if the weight of the body is 500 lbs., and the constant pulling force is 25 lbs., required the space over which the body will be moved from rest in 10 seconds.

In the same problem, if the constant frictional resistance is 10 lbs., what will be the velocity at the end of 100 feet?

In the fourth problem above, if a body weighs 100 lbs. at a place where $g = 32$ feet, and is placed in a hollow space at the centre of the earth; required the pulling force necessary to produce a velocity of 100 feet in 10 seconds.

)0. **Problems.**—1. *Suppose that a body is on a friction-s plane, and is moved horizontally by a weight attached it by means of a string pass-ȷ over a pulley; required ; space passed over by each ' the bodies in a time t, there ing no resistances from the ılley, string or air.*

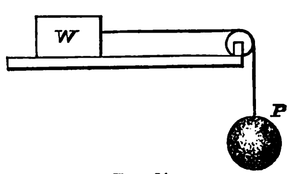


FIG. 21.

We have, $P =$ the effective moving force,

$$\frac{P + W}{g} = \text{the total mass moved};$$

ınce, from Article 86, we have

$$f = \frac{P}{\frac{P+W}{g}} = \frac{P}{P+W} g;$$

and this, in equation (4) of Article 24, gives

$$s = \frac{1}{2}\,\frac{P}{P+W}\,gt^2.$$

(REMARK.—If the student finds it difficult to distinguish between the moving force and the mass moved, let him imagine the whole system placed in the hollow sphere described in Article 76. Then both bodies, P and W, will be destitute of weight, and no motion can result from the action of P. Now conceive that a string is attached to the body P, and passed through the shell to some point on the surface where a man pulls with a constant force of P pounds; the result will be the same as that given in the preceding problem, for the man will be obliged to move the mass of both bodies. When the system is on the earth, gravity pulls with a force of P pounds on the body P, and nothing in the direction of motion of W; hence the force P must move both P and W.)

2. *Required the tension of the string in the preceding problem.*

Let $T =$ the tension.

Conceive the string to be severed, and a force applied equal to the tension T, pulling upward on the body; then will the *effective moving force be*

$(P - T)$ *pounds.*

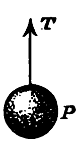


FIG. 22.

The mass of the body P, will be

$$\frac{P}{g};$$

hence, according to Article 86, we have

$$P - T = \frac{P}{g}\,f.$$

The *effective pulling force* on W, Fig. 21, is the tension of the string, hence, according to Article 86,

$$T = W\frac{f}{g}.$$

Eliminating f from these equations gives

$$T = \frac{WP}{W+P}.$$

3\. *A string passes over a wheel and has a weight P attached at one end and W at the other; if there are no resistances from the string, wheel, or air, and the string is devoid of weight, required the resulting motion.*

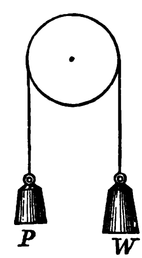


FIG. 23.

The *effective* moving force $F_{,} = (W - P)$ *pounds;*

the mass moved $= \dfrac{W+P}{g}$;

hence, from Article 86,

$$F = W - P = \frac{W+P}{g} f;$$

$$\therefore f = \frac{W-P}{W+P} g;$$

and, from Article 24, we have for the space

$$s = \tfrac{1}{2} \frac{W-P}{W+P} gt^2,$$

and for the velocity,

$$v = \frac{W-P}{W+P} gt.$$

(**These problems are *ideal*, since they discard certain elements, such as the mass of the pulley, friction, and stiffness of the cord. These, however, may all be computed, but they make the problem too complicated for this part of the work. The chief object here is to confine the attention to the relation between forces and the motion produced upon masses.**)

4. *Find the tension of the string in the preceding problem.*

$$Ans. \; T = \frac{2PW}{P+W}.$$

5. *A man, whose weight is W, stands on the platform of an elevator as it* descends *the vertical shaft of a mine; if the platform descends with a uniform acceleration of $\frac{1}{4}g$, show that his pressure upon the platform is $\frac{3}{4}W$. What will it be if the platform* ascends *with the same uniform acceleration?*

EXAMPLES.

1. In Fig. 21, if $P = W$, how far will the bodies move in 5 seconds?
2. In Fig. 23, if $W = 2P$, how far will the bodies move in 5 seconds?
3. In Fig. 23. $W = 50$ pounds; what must be the weight of P so that it will descend 10 ft. in 5 seconds? What, that it may ascend 10 ft. in 5 seconds?
4. In Fig. 23, if $P = 5$ ounces, and $W = 4\frac{1}{2}$ ounces, and it is observed that P descends 6.8 ft. in 2 seconds; required the value of g.
5. In Fig. 23, if $W = 10$ pounds, required the weight of P that the tension may be 1 pound; 5 pounds; 10 pounds; 20 pounds.
6. In Fig. 21, if $W = 20$ pounds, $P = 2$ pounds; required the time necessary for the bodies to move 10 ft.

EXERCISES.

1. A spring balance may be inserted in the string of the preceding problems in such a way as to indicate the tension. Suppose that such a balance were inserted in the vertical part of the string in Fig. 21, and the string cut off above it, what will the balance indicate afterwards?
2. If a man pulls a weight vertically upward by means of a cord attached to the body, will the tension on the cord equal or exceed the weight of the body;—the weight of the cord being neglected. Consider the case when the motion is accelerated, and when it is uniform.
3. If a man stands on the platform of an elevator as it descends a shaft with an acceleration, how will his pressure upon the floor compare with his weight? How if the platform is ascending?
4. If a man slides down a vertical rope, *checking* his velocity by grasping it more or less firmly, but with a constant grip, how will the tension on the rope compare with his weight?

CHAPTER III.

WORK—FRICTION.

91. Work *is the overcoming of resistance continually recurring along the path of motion.*—Thus a horse, while drawing a load on a cart, does work by constantly overcoming the friction of the axle and the resistances of the roadway. The same effort, however, may be exerted in producing motion only, producing *live* or *stored work.* Hence, the following is a more general definition: **Work** *is the effect produced by a force in moving its own point of application in such a way that it has a component motion in the direction of action of the force.*

92. Measure of Work.—A horse that draws a load two miles does twice the work of drawing it one mile, and one-fifth the work of drawing it ten miles. The work, therefore, varies directly as the space over which the resistance is overcome. It is also evident that, if the horse had drawn a load twice as large, he would have done twice the work in the same distance; hence, the work also varies directly as the resistance. This principle is general, and applies to all cases in which the force is constant.

Hence, if

$F =$ a constant force;
$s =$ the space over which F acts, and
$U =$ the work done by F;

then

$$U = Fs.$$

93. The work of a variable force is found by dividing the space into small parts, so small that the force

may be considered constant over each part, and taking the sum of all the elementary works.

Let

$F =$ the force acting over any one of the elementary spaces;

$\overline{\Delta s} =$ an elementary space;

then

$$U = \Sigma F\overline{\Delta s}.$$

94. *Mere motion is not work.*—If the planets move in space without meeting any resistance, they do no work.

95. The Unit of Work *is the raising of one pound of matter vertically one foot*, and is called a foot-pound. The resistance overcome by raising matter is the force of gravity. A weight of one pound drawn horizontally is not the unit, unless the frictional resistance should happen to equal the weight. It is not the weight moved, but the resistance overcome, that constitutes work.

96. Work represented by a Diagram.—When the force is constant the work may be represented by a rectangle.

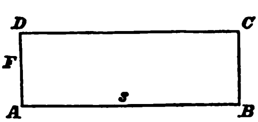


FIG. 24.

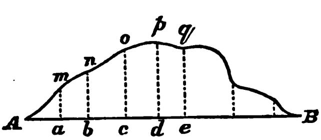


FIG. 25.

Thus, in Fig. 24, let the base of the rectangle represent the space s, and the altitude represent the force F, then will the *area of the rectangle be*

$$Fs,$$

which is the expression given in Article 92 for the work.

If the force does not follow a known law, we may still find the work approximately by constructing a curve, the abscissas of which, Aa, ab, bc, etc., shall represent the

spaces, and the corresponding ordinates, am, bn, co, etc., shall represent the corresponding resistances; then the area of the figure will represent the work, for the area will be $\Sigma F \overline{\Delta s}$. The area may be found to any degree of approximation by dividing it into an indefinite number of trapezoids, finding the area of each and taking their sum.

If the force varies directly as the space over which it acts, the work may be represented by the area of a triangle, of which the base represents the space and the altitude, the final force. For, any ordinate bc will be directly as its distance from A.

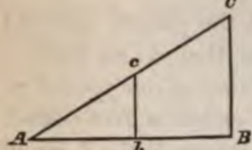


FIG. 26.

97. *The total work is independent of the time required to perform it.*—For, the space may be passed over in a longer or shorter time without affecting the product of F into s. The horse that draws a load one mile, does a definite amount of work, whether it be done in one hour or five hours.

98. **Time and velocity** *are implicitly implied in doing work;* for, the space involves both. If the space be passed over with a constant velocity we have

$$s = vt,$$

in which

$v =$ the velocity, and
$t =$ the time.

Hence,

$$U = Fs = Fvt;$$

therefore, if the velocity be uniform, the work will vary directly as the time; and if the time be constant, the work will vary directly as the velocity.

99. **Dynamic Effect.**—If the velocity be uniform and $t = 1$, we have

$$work = Fv.$$

This is the work done in a unit of time, and is called *Mechanical Power*, or *Dynamic Effect* and sometimes simply *Power*. It is the *rate* of doing work, or simply *work-rate.*

100. The unit of Dynamic Effect is called the horse-power. It is 33,000 pounds raised one foot per minute. It was determined by Boulton & Watt by means of trials with horses at a colliery in England. It is doubtless much larger than what the average of good horses can do for hours in succession, but it may be considered as an arbitrary unit, by which the work done by a horse or other working power in a *given time*, may be measured. It is used to measure the *efficiency* of hydraulic motors, steam and air engines and other machinery.

101. Work may be **useful or prejudicial.**—That is *useful* which produces the article sought, and that which wears out the machinery is *prejudicial.* The former produces money for the mechanic while the latter costs money.

Prejudicial work is generally frictional in its character, but all friction is not prejudicial. Thus, the friction between the driving wheels of a locomotive and the track is necessary, and hence, useful. In a similar way the friction between belts and pulleys is useful.

It is not always possible to distinguish between useful and prejudicial work. The latter always accompanies the former, but we know that, for economy, the latter should be reduced as much as possible.

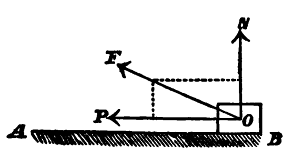


FIG. 27.

102. If the force acts at an angle *with the line of motion* it may be resolved into two forces, one of which

is parallel to the path described by the body and the other perpendicular to it. Thus, in Fig. 27, the normal component of the force F is

$$N = F \sin FOP;$$

and the horizontal component is

$$P = F \cos FOP.$$

There being no motion normal to the path, the former component does no work; and the work done by the latter, according to the definition, will be

$$U = Ps = Fs \cos FOP.$$

If a body is moved around a semi-circumference by a constant force acting parallel to a fixed diameter, the work will be the product of the force into the diameter.

103. Work in a Moving Body.—A moving body cannot be *instantly* brought to rest, and *when both resistance and space are involved in a result, work has been done.* The work which a moving body is capable of doing equals the product of the mean resistance which it overcomes into the space over which it works. Thus, if a cannon ball should penetrate the earth 10 feet, and the mean resistance were 500 pounds, the work done would be 5,000 foot-pounds. Another mode of measuring it is given in Article 111.

Friction.

104. Friction *is that force or resistance between two bodies which prevents, or tends to prevent, one body from being drawn upon another.*

All bodies are rough. However perfectly they may be polished, an inspection of their surfaces with a glass of high magnifying power, shows that they are still *very* rough. A *smooth surface* is a comparative term, implying

that it is more or less smooth. A *perfectly smooth* surface probably does not exist; but when the term is used it means that the surface offers no resistance of any kind. It is an *ideal* surface.

It is certain that if two perfectly smooth plane surfaces were brought in contact, they would offer a great resistance to being drawn upon one another on account of the adhesion between the bodies. They would hold to one another nearly as strongly as if they were solid. When we refer to *smooth surfaces* in problems, this force is also excluded.

105. Experiments in regard to Friction.—M. Morin, a French savant, was one of the first to determine the laws of friction. These laws were deduced by experiments upon a variety of substances under a variety of conditions. A device similar to that shown in Fig. 29 was used. A body W was placed upon a long strip of another body, and it was desired to determine the friction between them. A strong fine cord was attached to the body, and, passing over a pulley at the end of the platform, was attached to a dish in which were placed weights P. The weights P were made to exceed slightly the frictional resistance, and thus pull the body W along the other body. The space over which they moved in a given time was then observed, and with the data thus obtained the friction was computed as shown in Article 109.

Weights were added to the body W so as to produce greater pressure; also weights were added to the dish and taken from it, so as to produce a greater or less velocity.

Different substances were used for the body W, and also for the horizontal strip on which the body W slid.

106. Angle of Friction.—Suppose that a body W is placed on an inclined plane AC, one end of which is

gradually raised until motion begins, or the body is in state bordering on motion. The weight W may be resolve into components; one, parallel t the plane, which tends to pull th body down it, and the other pe pendicular to it.

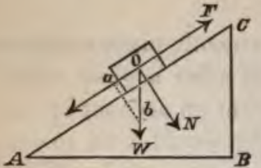


Fig. 28.

Draw Oa parallel to AC, an ab perpendicular to it; and le Ob *represent* W. Then, accordin to Article 52, we have

$$F = Oa = W \sin abO = W \sin CAB.$$

This equals the resistance due to friction.

Let

$$N = \text{the normal pressure} = ab;$$

then

$$N = W \cos abO = W \cos CAB;$$

hence,

$$\frac{F}{N} = \frac{W \sin CAB}{W \cos CAB} = \tan CAB;$$

which is called the *angle of friction*, or *angle of repose.*

107. Laws of Friction.—The following laws have bee deduced from the experiments of Morin and others:

1. Friction of motion is slightly less than that of rest.

2. The total amount of friction is independent of th extent of the surfaces in contact.

3. The amount of friction between two surfaces vari directly as the normal pressure, and with the character of the surfaces in contact.

4. Sliding friction is independent of the velocity.

These laws are sufficiently accurate for ordinary veloc ties when the surfaces do not *abrade*, or cut one anothe Lubricants are employed to diminish friction. Oil

the most common lubricant, though water is better in some cases.

108. Coefficient of Friction.—According to the third principle of the preceding article, it follows that *the ratio of the total friction to the total normal pressure between two surfaces is constant.* This ratio is called the *Coefficient of Friction.*

Let N = the normal pressure;
F = that force which is just sufficient to produce motion when acting parallel to the plane of the surfaces;
μ = the Coefficient of friction;

then

$$\mu = \frac{F}{N}.$$

Comparing this with the equation of Article 106, we see that *the coefficient of friction equals the tangent of the angle of repose.*

If the body moves on a horizontal plane, the normal pressure equals the weight, hence,

$$\mu = \frac{F}{W}.$$

If $W = 1$, $\mu = F$; hence, the coefficient of friction equals the friction caused by one pound of the body.

109. To find the value of the coefficient of friction of motion.

Let W = the weight of the body on the plane,
P = the weight which moves it,
f = the acceleration,
μ = the coefficient of friction;

then the total friction will be

$$\mu W,$$

and the *effective* moving force, neglecting the motion of the pulley and the cordage, will be

$$P - \mu W,$$

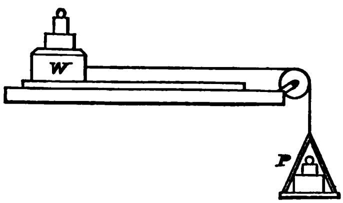


FIG. 29.

and the mass moved will be

$$(W + P) \div g;$$

hence, according to Article 86,

$$P - \mu W = \frac{P + W}{g} f;$$

$$\therefore f = \frac{P - \mu W}{P + W} g.$$

This value in equation (4), of Article 24, gives

$$s = \tfrac{1}{2} \frac{P - \mu W}{P + W} gt^2;$$

from which we find

$$\mu = \frac{gt^2 P - 2(P + W)s}{gt^2 W}.$$

Knowing the weights P and W, and measuring care fully the space s and time t, the second member becomes completely known; and hence, by reduction, μ becomes known.

For oak on oak, $\mu = 0.48$ when the fibres are parallel to the motion, and
0.19 when the fibres are perpendicular to the motion.

Wrought iron on cast iron, $\mu = 0.18$.
Cast iron on cast iron, $\mu = 0.15$.

M. Pambour made experiments upon the frictional resistance of trains of cars on some railroads in England, and

found the friction to be 8 pounds per ton gross; hence, the coefficient was

$$\mu = \frac{8}{2240} = \frac{1}{280}.$$

Experiments in this country gave $6\frac{1}{2}$ pounds per ton gross under favorable conditions.

110. Problems.—1. *A piston is moved in a horizontal cylinder, as shown in Problem 1, Article 89, by a constant steam pressure of F pounds. At what point must the pressure be instantly reversed so that the full length of the stroke shall be* a *inches, there being a constant friction of F_1 pounds throughout the stroke?*

The effective driving force will be $F - F_1$ *pounds.*

The effective stopping force will be $F + F_1$ *pounds.*

Let

s = the space over which the former acts, and

s_1 = the space over which the latter acts;

then

$$s + s_1 = a.$$

The *work done upon the piston* by the driving force will equal that done by the stopping force; hence,

$$(F - F_1)s = (F + F_1)s_1.$$

Eliminating s_1 between these equations, gives

$$s = \frac{F + F_1}{2F} a.$$

2. *A stream of water falls over a dam* h *feet high, and has a section of* a *square feet at the foot of the fall; required the horse-power constantly developed.*

(The section is taken at the foot of the fall, for the velocity is measured at that point.)

The velocity at the foot of the fall will be

$$v = \sqrt{2gh},$$

and the volume of water which passes over the dam in second will be

$$a\sqrt{2gh}.$$

The weight of a cubic foot of water being $62\frac{1}{2}$ pou the weight of the quantity will be

$$W = 62\tfrac{1}{2}a\sqrt{2gh} \textit{ pounds per second,}$$

$$= 60 \times 62\tfrac{1}{2}a\sqrt{2gh} \textit{ pounds per minute.}$$

The weight, multiplied by the height h, through w it falls, will give the work it can do in one minute, the result, divided by 33,000, will give the horse-powe

$$\therefore H^P = \frac{60 \times 62\frac{1}{2} \times (64\frac{1}{3})^{\frac{1}{2}} a h^{\frac{3}{2}}}{33000} = 0.9114ah^{\frac{3}{2}}.$$

3. *Find the work necessary to draw a body up an clined plane.*

In Fig. 28, Article 106, let

$l = AC$; $b = AB$; $h = BC$, $W =$ the weight of the b

As shown in Article 106, the normal pressure will l

$$N = W \cos CAB = \frac{b}{l} W,$$

and hence, the friction will be

$$\mu N = \mu \frac{b}{l} W;$$

which, multiplied by the length of the plane, gives work necessary to overcome the friction;

$$\mu N l = \mu b W.$$

The component of the weight along the plane will be

$$W \sin CAB = \frac{h}{l} W,$$

and the work of overcoming the weight will be l times this result, or

$$hW;$$

hence, the total work will be

$$\mu b W + h W;$$

hence, it equals the work which would be necessary to draw the body horizontally from A to B, and lift it vertically from B to C.

4. *Required the work necessary to compress a coiled spring a given amount.*

It is found by experiment that, as long as the elasticity of a spring remains perfect, the amount of compression varies directly as the compressing force. That is, if one pound compresses it one inch, two pounds will compress it two inches, and so on. Hence, if

p = the force which compresses a spring one inch,
P = the total compressing force,
s = the amount of the compression produced by P;

then

$$P = ps,$$

and the work is represented by Fig. 26, in which $AB = s$, $BC = P$.

$$\therefore U = \tfrac{1}{2} Ps = \tfrac{1}{2} ps^2.$$

EXAMPLES.

1. How many cubic feet of water will a 50 horse-power engine raise in an hour from a mine 500 feet deep, if a cubic foot of water weighs $62\frac{1}{2}$ pounds?
2. Find the work necessary to raise the material in making a well 20 feet deep and 3 feet in diameter, if the material weighs 140 pounds per cubic foot.
3. The pressure on a steam piston, moving horizontally, as in Prob. 1, Art. 89, is 1,000 lbs., the friction 200 lbs.; how far must the pressure act before it is instantly reversed that the full stroke may be 12 inches?
4. The French unit of work is one kilogramme raised vertically one metre; required the equivalent in foot-pounds. (Take the metre at 39.37 inches and the kilogramme at 2.2 pounds.)
5. According to Navier it requires 43,333 French units of work to saw a square metre of green oak; how many foot-pounds will be required to saw a square foot of the same material?
6. A stream of water falls vertically over a dam 12 feet high, and has a transverse section of one square foot at the foot of the fall; required the horse-power constantly developed.
7. A hammer, whose weight is 2,000 pounds, falls vertically 8 feet; how far will it drive a pile into the earth if the constant resistance is 10,000 pounds?
8. If it is found by means of a spring balance that a span of horses pull with a constant force of 200 pounds in drawing a plough; if they travel at the rate of 2 miles per hour, what will be the mechanical power required to work the plough?
9. In Fig. 29, let $W = 40$ pounds, $P = 8$ pounds, and it

is observed that P moves over 4 feet in 3 seconds; required the coefficient of friction.

10. In Fig. 29, $W = 25$ pounds, $P = 5$ pounds, and the coefficient of friction $= 0.15$; required the space over which the bodies will pass in 5 seconds.

EXERCISES.

1. What is the unit of work? Is work the same as force?
2. Is friction force? When is it useful and when prejudicial?
3. In what sense is work independent of the time, and under what circumstances is it dependent upon the time?
4. A body placed on a plane, which is elevated at an angle of 15 degrees, is just on the point of moving; required the coefficient of friction between the body and the plane.
5. A body, whose weight is 25 pounds, is on a horizontal plane; required the tension of a string by which the body is drawn along uniformly, the coefficient of friction being $\frac{1}{5}$.
6. Define mechanical power. Is it work?

CHAPTER IV.

ENERGY.

111. Energy is a term to express *the ability of an agent to do work.* We have seen, Article 103, that a moving body is capable of doing work. A slight consideration of bodies at rest shows that they are also *capable* of doing work. Thus, the water in a mill-pond is capable of doing useful work by being passed through a water-wheel. To do work the water must be in motion, but the weight of the water falling through a given height will do a certain amount of work, and this amount can be determined while the water is in *position.* Similarly, the same can be shown in regard to other bodies at rest. These ideas have given rise to the terms *Kinetic energy* and *Potential energy.*

112. Kinetic Energy *is the energy of a moving body,* and is the work which the body must do in being brought to rest. It is *visible energy.*

The work which a moving body is capable of doing is the same as if it had fallen in a vacuum through a height sufficient to produce the same velocity.

Let

$W =$ the weight of a body,
$v =$ its velocity,
$h =$ the height through which the body must fall to produce the velocity v.

The work necessary to raise a body a height h will be, according to Article 92,

$$Wh.$$

If the body fall freely through the same height, it will be capable of doing the same amount of work when it reaches the foot of the fall. The velocity produced in falling a height h will be (Article 72),

$$v = \sqrt{2gh}\,; \ \therefore h = \frac{v^2}{2g}.$$

Substitute this value in the preceding expression, and making,

$$M = \frac{W}{g} \text{ (Article 80)},$$

we have,

$$Work = Wh = W\frac{v^2}{2g} = \tfrac{1}{2}\frac{W}{g}v^2 = \tfrac{1}{2}Mv^2 = K.$$

The expression $\frac{1}{2}Mv^2$ is called the *kinetic energy*, and is represented by K, the initial letter of kinetic. It is also called the *vis viva*,* or living force of the body. Hence, *the kinetic energy of a moving body equals the work stored in it.*

113. Potential Energy *is latent energy.* It is the work which a body is capable of doing in passing from one *condition* or *position* to another. Thus, the power in a coiled spring is *potential;* but, when freed from its restraining power, it may move the wheels of a watch, or clock, or drive other machinery, in doing which it is changed from a condition of tension towards one free from tension. The

* Mv^2 is often called the *vis viva*, but its use is not quite as convenient as the definition in the text, and there is a growing tendency towards the general adoption of the definition given above. It makes no difference, however, which is used, provided it is always used in the same sense.

power in a weight held at a given height is *potential;* but, in descending, it may be made to turn machinery and thus do work, in doing which the body passes from a high position to a lower one. The power stored in coal, wood, or other fuel is *potential;* but, if the fuel be burned, it may generate steam and thus do work, in doing which it is changed from the condition of fuel to that of ashes, cinders, smoke, etc. The power in gunpowder is *latent;* but, if the powder be exploded, it will do work, and may be made to throw a cannon ball, or rend rocks, or produce motive power. The power contained in food is potential; but the food, by nourishing animals, and thus imparting strength to them, becomes a source of work. The power contained in zinc is potential; but the zinc, when acted upon by acids, becomes active and capable of driving electro-magnetic engines. Air compressed and stored in a vessel contains potential energy; but, by acting upon suitable machinery as it expands itself, it will do work. The power of steam confined in a boiler is potential; but, if the steam be passed through suitable mechanism, it will do work.

The expression, *Change of Position*, is sufficiently comprehensive to express all the changes of condition. *Potential Energy* has been defined as *Energy of Position*. It is represented by the Greek letter Π, the initial of potential.

Heat Energy.

114. General Statement.—It is well known that heat may be produced by friction, or rather, according to our present knowledge of the subject, *the work of overcoming friction produces heat.* Thus, the boy, by rubbing a brass button briskly on his sleeve, soon makes it too hot

for the comfort of his neighbor's hand. Rubbing two sticks together makes them warm. One of the earliest methods of obtaining heat was by friction. The heat produced by the friction of a match, when rubbed on a rough surface, ignites the phosphorus, which, by burning, so increases the heat as to set the wood on fire. Axle bearings in machinery often become so hot from friction as to set fire to the oil and wood which surround them. In an experiment made by Sir Humphrey Davy in 1799, two pieces of ice, rubbed together in vacuo at a temperature *below* 32° F., were melted by the heat developed at the surfaces of contact.

Iron and other substances may be heated by being struck rapidly. Compressing air, or other gaseous bodies, develops heat.

115. The Dynamic theory of heat rests upon the hypothesis that heat consists of the motion of the molecules of a body, or is the result of that motion, and that to produce these motions requires a definite amount of mechanical energy. This hypothesis is confirmed by the experiments of Joule, who produced heat in a variety of ways—compressing air, compressing gases, agitating water, and by friction between cast-iron surfaces.

According to Joule's experiments, if a body weighing one pound were permitted to fall freely 772 feet in a vacuum, and all the energy thus acquired could be utilized in heating one pound of water, it would raise the temperature 1° Fahrenheit. This is Joule's equivalent, and in the mathematical theory of heat is represented by J. It is recognized as *the* mechanical equivalent of heat.

116. The Mechanical Equivalent of Heat.—There is a definite relation between the work expended in overcoming friction and the heat which is produced by it.

The earliest experiments of which we have any historical knowledge tending to prove this fact were made by Count Rumford. In 1798 he published in the Trans. of the Royal Phil. Society some of his experiments upon this subject. A brass cannon weighing 113 pounds was revolved horizontally, at the rate of 32 revolutions per minute, against a blunt steel borer with a pressure of 10,000 pounds. In half an hour the temperature of the metal had risen from 60° to 130° F. This heat would have been sufficient to raise the temperature of five pounds of water from 32° to 212°. In another experiment the cannon was placed in a vessel of water and friction applied as before. In two hours and a half the water actually boiled. *The heat generated in this case was calculated* by Rumford to be at least equal to that given out, during the same time, by the burning of nine wax candles, three-quarter inch in diameter, each weighing 245 grains.

Fourrier, in the year 1807, gave the laws of the transmission of heat by radiation and conduction, and laid the foundation for the *mathematical theory of heat.* Sadi Carnot, in his work entitled "Réflexions sur la puissance motrice du feu," published in 1824, compared the energy of heat to that of a fall of water from one level to another.

This branch of science, however, made little or no progress until it was shown that there was a definite relation between heat and work. Several contemporaneous investigators entertained the view that such definite relation existed, and labored independently to prove it. In 1842 Dr. Mayer, a physician of Heilbronn, formally stated that there exists a connection between heat and work, and first introduced the expression, "*mechanical equivalent of heat.*" In the same year Colding, of Copenhagen, published experiments on the production of heat by friction,

from which he concluded that the quantity of heat produced by friction was directly proportional to the work expended.

But, the most important were the labors of Dr. J. P. Joule, of Manchester, England, who, during the years 1840 to 1843, by a series of very careful and elaborate experiments, determined a value for the mechanical equivalent of heat which is considered as the most reliable ever found. (See Phil. Trans., 1850, p. 61.)

117. Joule's Experiments.—It being impracticable to change the energy of a body falling freely into heat in such a way as to measure the exact equivalent, Joule resorted to different devices. One of the most reliable is the following:

A copper vessel *B*, Fig. 30, was filled with water and provided with a brass paddle-wheel (shown by dotted lines), which could be made to rotate about a vertical axis. The

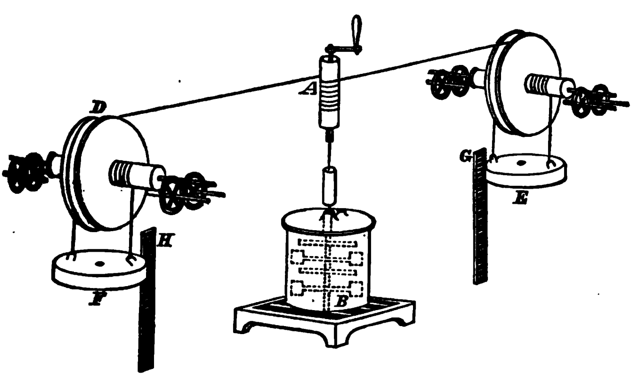


FIG. 30.

paddle had numerous openings, so as to agitate the water as much as possible when it rotated in the vessel. The vessel was closed so as to prevent the escape of the water, and was

provided with a thermometer to measure a change of the temperature. Two weights, E and F, were attached to cords which passed over the axles of the pulleys C and D, and were connected with the axis A, so that as the weights descend the paddle would be made to revolve. The height of the fall was indicated by the scales G and H, the total fall in Joule's experiments being about 63 feet. The roller A was so connected to the axis of the paddle that, by removing a pin, the weights could be wound up without disturbing the axle. In this way the experiments were repeated twenty times.

The work done by gravity was expressed by $(F+E)\times h$. This was all expended in the following ways: 1st, chiefly in overcoming the resistance of the water; 2d, in overcoming the friction of the several bearings; and lastly, the kinetic energy in the moving parts at the instant the motion was stopped. In the experiments the two latter were reduced as much as possible by the mechanical arrangements, but their effects were computed and deducted from the work done by gravity.

The remaining work was expended in overcoming the resistance of the water, and thus developed heat.

Taking as a unit of heat that necessary to raise the temperature of one pound of water 1° F., *the unit of heat*
8 *equals* 772 *foot-pounds of work.*

If the unit of heat be that necessary to raise one kilogramme of water 1° C., then *a unit of heat equals* 424 *kilogramme-metres.*

General Principles of Energy.

118. Transmutation of Energy.—We have seen that kinetic energy may be changed into an equivalent of heat

energy through the agency of friction. We also know that heat energy may be changed into kinetic energy, as is constantly done in the ordinary steam-engine. Other energies are constantly brought into action, such as electricity, magnetism, chemical action, forces of polarity, and indeed any agency by which matter is moved. The relations between the different kinds of energies are not well known, except that between heat and visible energy; but it is believed that they all consist of some kind of molecular motion, and are *kinetic* or *potential* according to circumstances, and that they are changeable one into the other.

This transmutation is constantly going on. To illustrate it with an example: The energy of the sun's heat is stored in the plant and there becomes potential. The plant may be changed to coal, imparting some of its energy to surrounding objects, but concentrating its remaining potential energy into a smaller space. Coal is raised from its bed by means of kinetic energies, and used in a locomotive engine, for instance, where it is burned and becomes kinetic. One portion of this energy becomes stored in the steam in the boiler; another portion escapes with the smoke; still another portion escapes through the walls of the fire-box; and another is thrown away with the live coals which fall through the grate, or are hauled out of the door of the fire-box.

The steam from the boiler is admitted into the cylinder of the engine, and a portion of this energy is utilized in driving the piston, and another portion escapes with the exhaust steam. The energy of the piston is expended in producing heat on the track, heating the axles under the cars, overcoming the resistance of the air, wearing the couplings between the cars and the working parts of the

machinery, bruising the ends of the rails, crushing the ties under the track, disturbing the earth or other material which forms the roadway, and imparting kinetic energy to the train. The energies which are transmitted to these several elements finally disappear in heat, which is quickly absorbed by the atmosphere, beyond which we are unable to trace it with any degree of certainty.

119. Conservation of Energy.—These principles are included in a general law called *the Conservation of Energy*, which may be stated as follows:

1. *The total amount of energy in the Universe is constant.*

2. *The various forms of energy may be converted the one into the other.*

It follows from the former of these that no energy is ever lost. *Energy is indestructible.*

This law is the result of a long series of observations, experiments, and generalizations, but it is now considered as firmly established as any law in nature. It is accepted as a *fundamental law* in Physical Science, and is as universal in its application as the law of gravitation.

120. Non-equilibrium of Energies.—Energies *work* only as they pass from one condition to another, and this is done only when they are not in equilibrium. Thus, if two metals are equally hot, one cannot impart heat to the other. A body by losing heat loses energy. When steam works by expansion, the temperature is reduced. The energy stored in coal is developed by being burned, but the heat thus produced exceeds that which it imparts to steam or to other bodies. We know of no means by which heat energy can be changed into an equivalent of kinetic energy; for in every attempt to accomplish it there is an apparent loss of energy, some of it becoming

potential, or assuming an energy of a lower form. This is called *dispersion of energy*. We have, however, seen that visible energy is readily changed into heat. Should all the energies of the universe finally become changed into heat *uniformly distributed* throughout space, all motion would cease, and the universe would become virtually dead. Such a result has been predicted by some writers upon this subject.

121. Perpetual Motion.—By perpetual motion, in a popular sense, is not meant ceaseless motion, such as we see in the earth and other planets, but a *machine* which, when put in motion, will continue in motion indefinitely, without the application of additional power. But every machine, when in motion, must overcome the resistance of the air and the friction of the bearings, and as these resistances cannot be entirely removed, or annihilated, it necessarily does work, thus consuming the energy imparted to it. When all the energy has been consumed the machine will stop and remain at rest. *A perpetual motion machine is an impossibility.* In order to be possible it must expend more power than is imparted to it; or in other words, it must possess a self-creating power.

EXAMPLES.

1. How many foot-pounds of work is stored in a body which weighs 25 pounds, and has a velocity of 100 feet per minute?
2. If a body, moving with a velocity of 5 feet per second, penetrates the earth 2 feet, how far would the same body penetrate it moving with a velocity of 15 feet per second, the resistance of the earth being uniform along the path of the body in both cases?

3. If a train of cars weighs 60 tons, and moves at the rate of 40 miles per hour, how far will it move before being brought to rest by friction, the friction being 8 pounds per ton, no allowance being made for the resistance of the air?

4. A train of cars weighing 200,000 lbs., and moving at the rate of 20 miles per hour, is suddenly stopped; if all its energy be utilized in heating water, how many pounds of water would be raised in temperature from 32° F. to the boiling point, or to 212° F.?

5. In Fig. 29, Article 109, if $W = 10$ lbs., $P = 4$ lbs., $\mu = 0.2$, $g = 32\frac{1}{6}$, and the weight W is drawn 5 feet by the weight P falling the same distance, when the latter strikes the ground; how far will the weight W move before being brought to rest by friction?

6. If a piece of iron whose weight is 200 lbs. is moved at a uniform rate to and fro on another piece of iron, the coefficient of friction between them being 0.2, what must be the velocity of the moving body so that the heat developed by the friction would, if entirely utilized, raise the temperature of 5 lbs. of water 50° F. every 3 minutes.

EXERCISES.

1. Is force the same as energy?
2. Does kinetic energy mean work done, or ability to do work?
3. An animal eats food which, mechanically speaking, is potential energy; indicate some of the energies which follow the digestion of the food.
4. Why is not wood heated as much by a ball fired into it as iron is by a ball fired against it?
5. Will a ball flying through the air be warmed on account of the friction of the air?

6. What becomes of the energy due to the motion of the water in rivers?
7. Why do a person's hands become warm by rubbing them against one another?
8. If a ton of coal, which costs 5 dollars, will evaporate a certain amount of water, and it requires two cords of wood to evaporate the same amount, what will the wood be worth per cord for the same purpose?

CHAPTER V.

MOMENTUM.

122. The equation of Article 86 is

$$F = Mf.$$

Equation (1) of Article 24 is

$$v = ft;$$

and, eliminating f from these equations, gives

$$Ft = Mv.$$

The quantity Mv is called *the momentum* of a body whose mass is M. This, however, is merely giving a name to an expression, but by comparing it with the first member of the equation we see that it is *the effect which a constant force F produces in a time t.* We therefore call it a *time-effect*, and represent its value by Q.

$$\therefore\ Q = Mv. \quad . \quad . \quad . \qquad (1)$$

If a body has a velocity v_0 when the force begins to act, and a velocity v after a time t, then

$$Q = Ft = M(v - v_0). \quad . \quad . \qquad (2)$$

123. Momentum of a variable force.—If the force be variable, suppose that the time is divided into such small portions that the force may be considered as constant during each portion, then will the total effect be the sum of all the elementary momenta.

Let $\overline{\Delta t}$ = an element of time,

F = the force during any element of time;

then

$$Q = \Sigma F . \overline{\Delta t} = M(v - v_0).$$

In these equations F is the *effective* moving force.

124. Impulse.—*An impulse is the time effect of a blow.* When one body strikes another, as a hammer striking an anvil, its *effect* is an impulse. The body struck may be fixed or free to move; but the problems which are usually considered under this head generally pertain to those in which the body considered is free to move. Although the effect will be produced in an exceedingly short time, yet the result is a time-effect, and is measured in the same way as any other time-effect. In the case of a blow, the pressure between the bodies will be variable during contact; but for the sake of making a practical formula, let

F' = the *mean* pressure between the bodies,

M = the mass of the body considered,

v = the velocity produced by the blow;

then

$$Q = F't = Mv.$$

125. *Momentum is not a force.*—The force F is only one of the elements of the expression. The unit of momentum is one pound of mass moving with a velocity of one foot per second.

Momentum is sometimes called *quantity of motion* and sometimes *quantity of velocity*, but neither fully expresses its meaning. The expression *time-effect* is partly descriptive of its meaning, and is preferred to either of the others.

126. Instantaneous Force.—When an effect is produced in an imperceptibly short time, the agency which

produces it is sometimes called *an instantaneous force*, a term which *implies* that the effect is produced instantly, requiring no time for its action. *No force produces its effect instantly*, and hence the term is liable to mislead. Sometimes it is called *an impulsive force*, but this is also objectionable, for it implies that the effect is a force whereas the *effect* is either momentum or work. On account of these objections it appears advisable to use the term *impulse* instead of either of the above.

127. Problems.—1. *If a body, whose weight is 25 lbs., is drawn along a horizontal plane by a constant pull of 6 lbs., the coefficient of friction being* $\frac{1}{10}$, *what will be the velocity at the end of* t *seconds?*

The frictional resistance will be

$$\tfrac{1}{10} \text{ of } 25 \text{ lbs.} = 2.5 \text{ lbs.};$$

hence, the *effective* pulling force will be

$$F = 6 - 2.5 = 3.5 \; lbs.$$

$$\therefore\ v = \frac{Ft}{M} = \frac{3.5 \times 193}{25 \times 6}\, t = 4\tfrac{1}{2}\, t \; ft. \; per \; sec.$$

2. *Required the constant force necessary to impart to a body, whose weight is* 100 *lbs., a velocity* v *during* 5 *seconds.*

$$F = \frac{Mv}{t} = \frac{100}{5 \times 32\frac{1}{6}}\, v = \tfrac{5}{8}\, v \text{ nearly.}$$

Impact.

128. Impact *is the impinging of one body against another.*

It is *direct* when the line of motion of the impinging body is normal to the body struck, as in Figs. 31 and 32.

It is *central* when the line of motion passes through the centre of the body struck, as in Figs. 31 and 33.

It is *direct and central* when the line of motion passes through the centre of the body struck, and is normal to the surface at the point of impact, as in Fig. 31.

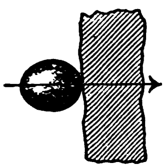


Fig. 31.

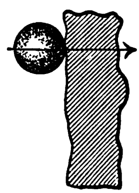


Fig. 32.

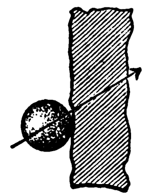


Fig. 33.

It is *oblique* when the line of motion is inclined to surface at the point of contact.

It is *eccentric* when the line of motion is normal to the surface of the body struck at the point of contact, but does not pass through the centre of the body, as in Fig. 32.

Elasticity.

129. Elasticity *is that property of bodies by which they gain, or seek to gain, their original form after they have been elongated, compressed, twisted, bent, or distorted in any way.* In order to discuss problems involving impact, it is necessary to know the laws of elasticity. It is found by experiment that, if a body be pulled by a force in the direction of its length, as in Fig. 34, it will be elongated; and if the elongation be small it will shorten itself when the pulling force is removed. It is found that all known substances are more or less elastic. Air and gases are highly elastic.

If bodies regain all their distortion after the force is

removed, they are called *perfectly elastic.* If they regain only a part of their distortion, they are called *imperfectly elastic,* and if they regain none of their distortion, they are called *non-elastic.* The *perfect* elasticity of *solids,* even within small limits, has been questioned, but for practical purposes many of them, such as steel, good iron, glass, ivory, etc., may be considered as perfectly elastic. Prismatic bars of good iron and steel may be elongated about $\frac{1}{1000}$ of their length without damaging their elasticity.

130. Coefficient of Elasticity.—Experiments show that the elongation or compression of prismatic bars of a solid, within small limits, varies directly as the pulling or pushing force, and inversely as the transverse section.

The coefficient of elasticity is the pulling force per unit of section divided by the elongation per unit of length.

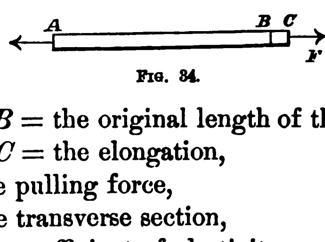


FIG. 34.

If $l = AB =$ the original length of the piece,
$\lambda = BC =$ the elongation,
$F =$ the pulling force,
$K =$ the transverse section,
$E =$ the coefficient of elasticity;

then

$$\frac{F}{K} = \text{the strain on a unit of section, and}$$

$$\frac{\lambda}{l} = \text{the elongation per unit of length;}$$

and, according to the above definition, we have

$$\frac{F}{K} \div \frac{\lambda}{l} = \frac{Fl}{K\lambda} = constant = E.$$

Values of E in pounds per square inch.

For wrought iron from	23,000,000 lbs. to 28,000,000 lbs.
For steel from	25,000,000 lbs. to 31,000,000 lbs.
For wood from	1,000,000 lbs. to 2,000,000 lbs.

131. Elongation of a prismatic bar.—Solving the equation in the preceding article, we have

$$\lambda = \frac{Fl}{EK},$$

which is the expression sought. This expression is true only for such strains as do not damage the elasticity.

132. Modulus (or Coefficient) of Restitution.—If an elastic body impinge upon another, the bodies will at first compress one another, the compression increasing rapidly for a very short time until a maximum is reached; after which, by virtue of their elasticities, they tend to regain their original form, and thus force themselves apart. The force which causes them to separate can act only while they are in contact with one another, but they may continue to regain their form after separation.

The force between the bodies during compression, and also during restitution, is constantly changing, and the law of change may be very complex. The *effect* is to constantly, but very rapidly, change the velocity of one or both the bodies during contact. If the bodies are composed of the same substance, and the impact is not so severe as to damage their elasticity, *the velocity gained by a body during restitution, divided by the velocity lost during compression, will be constant, and is called the modulus of restitution*, or *index of elasticity*. It is represented by e. If the bodies are moving in the same direction, the body struck will gain velocity during compression, and still

more velocity during restitution. In this case the *modulus of restitution* will be found by dividing the velocity gained during the restitution by the velocity gained during the compression. In short, it is the ratio of the effect on the velocity due to restitution to that due to compression.

According to these principles a simple expression may be found for e by assuming that the body struck is at rest, and so large that it will be unaffected by a blow from the moving body; for, in this case, the velocity at the instant of greatest compression will be zero, and hence the velocity lost during compression will equal the velocity of approach of the striking body, and the velocity regained will be that with which it rebounds.

Let

$v =$ the velocity of the impinging body at the instant impact begins;

$v_1 =$ the velocity at the end of the impact;

then,

$$e = \frac{v_1}{v} \quad . \quad . \quad . \quad . \quad (1)$$

Let the moving body *fall* upon the immovable one, and let

$H =$ the height of the fall,

$h =$ the height of the rebound;

then

$$v = \sqrt{2gH}, \qquad v_1 = \sqrt{2gh},$$

$$\therefore e = \frac{v_1}{v} = \sqrt{\frac{h}{H}}. \quad . \quad . \quad (2)$$

In this way the values of e have been found for the following substances:

Substance.	Value of e.	Substance.	Value of e.
Glass..............	0.94	Steel, soft.........	0.67
Hard baked clay....	0 89	Cork..............	0 65
Ivory..............	0 81	Brass.............	0.41
Limestone	0.79	Lead..............	0.20
Steel, hardened.....	0.79	Clay, just yielding	
Cast-iron...........	0.73	to the hand......	0.17

For perfectly non-elastic bodies e is zero. If the restitution were perfect during contact e would be unity, and the bodies would be called *perfectly elastic.* The *momentum lost* by a body during compression is generally called *the force of compression*, and that gained during restitution *the force of restitution.*

133. Problem.—*Given the masses and velocities of two perfectly free non-elastic bodies before impact, to find their velocities after impact.*

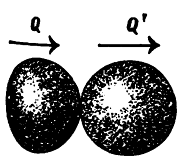


FIG. 35.

Take the simplest case, that of direct central impact. Let the body Q impinge upon Q', the motion being in the same direction.

Let

$M =$ the mass of Q,
$v =$ the velocity of Q before impact,
M' and v' the corresponding quantities for Q', and
$V =$ the common velocity after impact.

The momentum lost by Q during impact will be

$$Q = M(v - V);$$

and that gained by Q' will be

$$Q' = M'(V - v');$$

which, being *time-effects*, will equal one another; hence,

$$M(v - V) = M'(V - v')$$

$$\therefore V = \frac{Mv + M'v'}{M + M'}. \quad . \quad . \quad (1)$$

Clearing of fractions gives

$$(M + M')V = Mv + M'v'; \quad . \quad . \quad (2)$$

that is, the *momentum of the bodies after impact equals the sum of momenta before impact.*

To find the velocity in terms of the weights, substitute $\frac{W}{g}$ for M and $\frac{W'}{g}$ for M', in equation (1), and we have

$$V = \frac{Wv + W'v'}{W + W'}. \quad . \quad . \quad (3)$$

If the bodies are moving toward one another before impact one of the velocities will be negative. Making v' negative, we have

$$V = \frac{Mv - M'v'}{M + M'}; \quad . \quad . \quad . \quad (4)$$

in which if $Mv = M'v'$, we have $V = 0$; that is, *if two inelastic bodies of equal momenta impinge directly upon one another from opposite directions they will be brought to rest by the impact.*

134. Loss of velocity.—From equation (1) of the preceding article we find, by subtracting both members from v,

$$v - V = \frac{M'}{M + M'}(v - v'); \quad . \quad . \quad (1)$$

and similarly, subtracting both members from v' gives,

$$v' - V = -\frac{M}{M+M'}(v - v'); \quad . \quad . \quad (2)$$

which, being negative, indicates that there is a *gain* of velocity.

135. Impact of perfectly elastic bodies. — It has been observed in Article 132 that at the instant of greatest compression the bodies have a common velocity; hence, *at that instant*, the velocity will be given by the equations of Article 133; and the loss of velocity up to that instant will be given by the equations in Article 134. But if the bodies are *perfectly elastic, the effect upon the velocities during restitution will be exactly the same as during compression;* hence, the final loss of velocity will be double that during compression; therefore, for the striking body, it will be

$$\frac{2M'}{M+M'}(v - v'); \quad . \quad . \quad . \quad (1)$$

and for the body struck it will be

$$-\frac{2M}{M+M'}(v - v'). \quad . \quad . \quad . \quad (2)$$

These subtracted from the velocity before impact will give the final velocity.

Let

v_1 = the velocity of the former body after impact,

and

v'_1 = the velocity of the latter after impact;

then,

$$v_1 = v - \frac{2M'}{M+M'}(v - v'); \quad . \quad . \quad (3)$$

$$v_1' = v' + \frac{2M}{M + M'}(v - v'); \quad . \quad . \quad (4)$$

and these may be reduced to

$$v_1 = \frac{Mv + M'v'}{M + M'} - \frac{M'}{M + M'}(v - v'); \quad (5)$$

$$v_1' = \frac{Mv + M'v'}{M + M'} + \frac{M}{M + M'}(v - v'); \quad (6)$$

which, by means of equation (1) of Article 133, become

$$v_1 = V - \frac{M'}{M + M'}(v - v'); \quad . \quad . \quad (7)$$

$$v_1' = V + \frac{M}{M + M'}(v - v'); \quad . \quad . \quad (8)$$

which show, as they should, that the final velocity equals the common velocity at the instant of greatest compression, increased or diminished, as the case may be, by the velocity due to restitution.

136. Discussions of Equations (3) and (4) *of the preceding article.*

1st. Let $M = M'$, then we have

$$v_1 = v - (v - v') = v';$$
$$v_1' = v' + (v - v') = v;$$

that is, they will interchange velocities.

2d. Let $v' = 0$ and $M = M'$, then

$$v_1 = 0;$$
$$v_1' = v;$$

that is, the first body will be brought to rest and the second will take the velocity which the first had.

3d. Let the bodies be of equal masses, and move in opposite directions, then $M = M'$ and v' will be negative, and we have

$$v_1 = -v';$$
$$v_1' = v.$$

137. Impact of Imperfectly Elastic Bodies.—When the bodies are imperfectly elastic the force of restitution will be only the e^{th} part of the force of compression; hence, the velocity due to restitution will be the e^{th} part of expressions (1) and (2) of Article 134, or

$$\frac{eM'}{M + M'}(v - v');$$

$$-\frac{eM}{M + M'}(v - v');$$

and these, subtracted from the common velocity at the instant of greatest compression, give the final velocity. The results will be the same as equations (5) and (6) of Article 135, after the last terms of those equations have been multiplied by e; hence, we have

$$v_1 = \frac{Mv + M'v'}{M + M'} - \frac{eM'}{M + M'}(v - v'); \quad . \quad (1)$$

$$v_1' = \frac{Mv + M'v'}{M + M'} + \frac{eM}{M + M'}(v - v'). \quad . \quad (2)$$

Let the body M' be indefinitely large and at rest, then, making $M' = \infty$ and $v' = 0$, we have

$$v_1 = -ev;$$
$$v_1' = 0;$$

hence, the former body will rebound, and by disregarding the signs, using the numerical values only, we have

$$e = \frac{v_1}{v};$$

which is the same as equation (1) of Article 132.

Multiplying equation (1) by M and (2) by M', and adding the results, gives

$$Mv_1 + M'v'_1 = Mv + M'v';$$

and, since the index of elasticity has disappeared, the total momentum of the bodies before impact will be the same as after impact; or, in other words, *the total momentum of a free system remains constant.*

133. Loss of Kinetic Energy due to Impact.—The total kinetic energy of both bodies, before impact, will be

$$\tfrac{1}{2}Mv^2 + \tfrac{1}{2}M'v'^2;$$

and is independent of the directions of the movement of the bodies. After impact the kinetic energy will be

$$\tfrac{1}{2}Mv_1^2 + \tfrac{1}{2}M'v'^2_1;$$

in which substitute the values of v_1 and v'_1 from the preceding article, and we find

$$Mv_1^2 + M'v'^2_1 = Mv^2 + M'v'^2 - \frac{(1-e^2)MM'}{M+M'}(v-v')^2.$$

Since e^2 is always less than unity for solid bodies, $1-e^2$ will be positive, and the last term must be subtracted from the preceding; hence, practically, *in the impact of solid bodies there is always loss of kinetic energy.* The loss will be equivalent to the heat developed by the impact.

If the restitution were perfect, we would have $e = 1$, and the expression would become

$$\tfrac{1}{2}Mv^2 + \tfrac{1}{2}M'v'^2_1 = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}M'v'^2;$$

nce, *in the impact of perfectly elastic bodies, no energy lost.* This result shows the great utility of springs der carriages, carts, cars, etc., when they are drawn over ıgh roadways. A horse will do more useful work by ıwing loads upon a cart, the body of which is supported springs, than if the cart were unprovided with springs, l a locomotive will consume less coal in hauling a train cars properly mounted on springs than it would if there re no springs under them.

Γo find the loss of kinetic energy for perfectly non-stic bodies, make $e = 0$ in the above equation.

EXAMPLES.

[f a body, whose weight is 20 lbs., is pulled by a constant force of 5 lbs. for 5 seconds; required the momentum produced.

A prismatic bar of iron, whose section is 0.75 of a square inch, length 10 feet, coefficient of elasticity 26,000,000 lbs., is stretched by a pull of 9,000 lbs., what will be the elongation ?

A cylindrical bar of iron, whose diameter is $\frac{1}{2}$ inch, length 2 feet, is elongated $\frac{1}{80}$ of an inch by a pull of 2,500 lbs.; required the coefficient of elasticity.

Γwo perfectly non-elastic bodies, whose weights are 10 and 8 lbs., and velocities 12 and 15 feet per second respectively, moving in opposite directions, impinge upon each other; required their common velocity after impact.

A ball, weighing 20 lbs., moving with a velocity of 100 feet per second, overtakes a ball weighing 50 lbs., moving with a velocity of 40 feet per second, their modulus of restitution being $\frac{1}{2}$; required their velocities after impact. *Ans.* $35\frac{5}{7}$, and $65\frac{5}{7}$ feet.

6. In the preceding example, suppose the second body to be at rest; required the velocities after impact.

Ans. $-7\frac{1}{7}$, and $+42\frac{6}{7}$ feet.

7. In Example 5, suppose that they move in opposite directions with the velocities given; required the velocities after impact.

Ans. -50, and $+20$ feet.

8. A body falls from a height h upon a *fixed* plane of the same substance and rebounds; the modulus of restitution being e, required the whole distance it will move in being brought to rest.

Ans. $\dfrac{1+e^2}{1-e^2}h$.

9. In the preceding example find the whole distance when $e = 1, \frac{1}{2}, \frac{1}{4}$ or 0.

10. A body impinges upon an equal body at rest; show that the kinetic energy before impact cannot exceed twice the kinetic energy of the system after impact.

EXERCISES.

1. Is elasticity a force?
2. If W is the weight of a body and v its velocity, is Wv the momentum?
3. Does momentum enable one to determine the amount of resistance which a moving body may overcome?
4. Suppose that it is required to determine how far a ball would penetrate a body if fired into it; would the solution be effected by the principles of momentum, work, energy, vis viva, elasticity, or by simple force?
5. If two bodies move along *rough* surfaces, and finally impinge upon each other, will the formulas of Articles 133 and 135 enable one to determine the velocities after impact?
6. Can two perfectly non-elastic bodies of unequal masses approach each other with such velocities as to destroy their motions?
7. Can two perfectly elastic bodies have such relative masses and velocities that they will mutually destroy each other's motion by the impact of one upon the other?

. Is there any relation between the *coefficient of elasticity* and the *modulus of restitution?*

. Explain how the use of springs may prolong the life of cars, and also the track on railways.

Force, Energy, Work, Momentum.

139. The office which these several elements perform in he solution of problems is best shown by an example. uppose that two imperfectly elastic bodies impinge one pon the other. That which gives them motion is *force.* hat which determines their velocity after impact is *omentum.* That which determines their ability to compress, or break, or damage each other is *energy*, and the ompression is *work done.* The permanent compression maining in the bodies represents kinetic energy lost, hich has passed into an equivalent amount of heat.

CHAPTER VI.

COMPOSITION AND RESOLUTION OF PRESSURES.

140. Remark.—A force which acts upon a body w
out producing motion results in *pressure* only. If sev
pressures concur they may evidently be replaced b
single pressure which will produce the same effect as
combined effort of all the pressures. The single press
is called **the resultant pressure.** It might, perhaps,
assumed that the value of the resultant for statical pressu
is the same as for those pressures which produce moti
but many think that it is best to deduce the value for
former independently of the latter, as has been done in
following articles. We will find, however, that the resu
ant of mere pressures is the same as given in Arts. 52 to

141. If a pressure acts directly opposed to the result
of all the other forces in the system, and of the same inte
sity as the resultant, the system will be in equilibrium. T
resultant of an equilibrated system is zero. Since, in sta
cal problems, there is always equilibrium, *any one of t
forces reversed will be the resultant of all the others.*

142. Component Pressures.—In Fig. 36, if R is t
resultant of the pressures F and
the latter are called *component pr
sures.* If the resultant is
forces may be com
nents. There
component

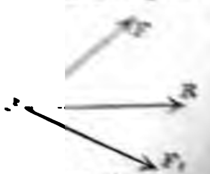


FIG. 36.

upon a particle.

143. *If two equal*

direction of the resultant will bisect the angle between them.

For no reason can be assigned why it should be nearer one than the other of the components.

144. *If three equal pressures act upon a particle, making angles of* 120° *with each other, they will be in equilibrium.*

For no reason can be assigned why any one should prevail over the other two.

145. Parallelogram of Pressures.—*If two pressures, acting on a particle, be represented in magnitude and direction by two straight lines drawn* FROM *the particle, and a parallelogram be constructed on these lines as adjacent sides, the resultant pressure will be represented in magnitude and direction by that diagonal of the parallelogram which passes through the particle.*

The proof of this proposition is given in two parts:—first, the *direction* of the resultant, and second, its *magnitude*. The following is Duchayla's proof:

146. Direction of the Resultant of Two Pressures.—*First*, let the forces be commensurable.

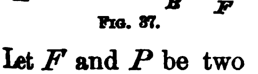


FIG. 37.

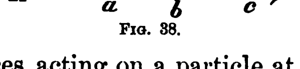


FIG. 38.

Let F and P be two forces acting on a particle at A; and Ac, Fig. 38, represent F and Ah represent P. Let Aa be the common measure of the forces; $F = 3Aa$, and $P = 2Aa = 2Ad$. On Aa and Ad construct a parallelogram, which, in this case, will be a rhombus, and the direction of the resultant will be the diagonal Ae, for it

bisects the angle dAa. Since a force may be considered as acting at any point in its line of action, the resultant may be considered as acting at e, and the force Aa becomes transferred to de, parallel to Aa, and Ad to the line ae. Combining the forces de and dh in the same manner, their resultant will be along the line di, and the three forces, Aa, Ad, dh, may be considered as acting at i; hence, Ai is the direction of the resultant of these forces. Now, combining ae (one-half of ai) with ab, gives af; and ef with ei gives ej, and the four forces, Aa, ab, Ad, dh, become transferred to j; hence, Aj is the direction of their resultant. Proceeding in this way, we find the resultant of Ac and Ah to be in the direction of AB, the diagonal of the parallelogram $AcBh$. Similarly, the proposition will be true for mP and nF, in which m and n are integers.

Secondly, let the forces be incommensurable. A ratio may always be found which shall differ from the true ratio by less than any assignable quantity, and hence the diagonal AB of the commensurable part will differ from the direction of the resultant by less than any assignable quantity. But no reason can be assigned why this diagonal should be on one side of the resultant rather than on the other; hence, they will coincide.

147. THE MAGNITUDE *of the resultant equals the length of the diagonal of the parallelogram, of which the adjacent sides represent the forces.*

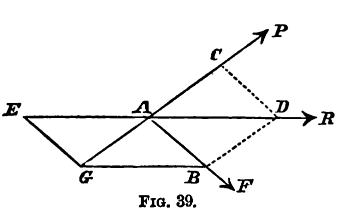


FIG. 39.

Let AB and AC represent in magnitude and direction the respective forces, and AD the *direction* of their resultant. Take

AE on AD produced backward, and of such length as to represent the *magnitude* of the resultant on the same scale as the forces P and F. Then, the forces AB, AC, and AE will equilibrate. On AE and AB, as adjacent sides, construct the parallelogram $ABGE$, and the diagonal AG will be the *direction* of the resultant of AE and AB.

Hence, AC must be in the same straight line as AG, and $AGBD$ will be a parallelogram; therefore $AD = GB$. But $BG = AE$; $\therefore AE = AD$; hence, the resultant of AC and AB equals AD in *magnitude*.

In Fig. 39 we have

$$AD^2 = AB^2 + AC^2 + 2AB \,.\, AC \cos BAC,$$

or

$$R^2 = F^2 + P^2 + 2FP \cos (F, P).$$

This equation is of the same form as those in Articles 14 and 56.

148. Triangle of Pressures. — *If three concurrent forces are in equilibrium they may be represented in magnitude and direction of action by the three sides of a triangle taken in their order.*

In Fig. 39, if EA is a force equal and opposite to the resultant of the forces P and F, and all three act upon a particle at A, they will be in equilibrium; and, according to the preceding Article, if GA represent P in magnitude, and the direction be from G towards A, and AB represent F in magnitude and direction, then will BG represent the equilibrating force AE.

It should be specially noticed that the sides of the triangle do not represent the *position* of the forces.

149. Conversely, *if three forces, acting upon a particle, are represented in magnitude and direction by the*

three sides of a triangle, TAKEN IN ORDER, *they will keep the particle in equilibrium.*

150. *If three forces, acting upon a particle, keep it in equilibrium, they will be proportional to the sines of the angles between the other two.*

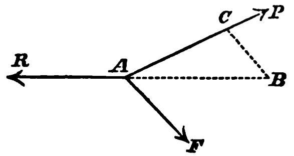


FIG. 40.

Thus, from Fig. 40, if the forces P, F, R, acting on a particle at A, keep it in equilibrium, we have

$$P : F : R = AC : CB : AB,$$
$$= \sin ABC : \sin CAB : \sin ACB,$$
$$= \sin (F, R) : \sin (P, R) : \sin (P, F).$$

151. Proposition.—*If the directions of three forces in equilibrium are given, and the magnitude of one is also given, the magnitudes of the other two may be found.*

The truth of this follows directly from the triangle of forces.

Generally, *if three forces, P, F, R, acting on a particle, keep it in equilibrium, if any three of the quantities P, F, R, and the angles which they make with each other are given, the remaining three quantities may be found,* PROVIDED *one of the given quantities is a force.*

For, the solution consists simply in solving a plane triangle, in which the given parts are a side and any two of the remaining parts of the triangle.

152. *If three forces, acting on a particle, keep it in equilibrium, they will be proportional respectively to the sides of a triangle formed by drawing lines perpendicular to the directions of the action-lines of the forces.*

For, a triangle thus formed will be *similar* to the triangle of equilibrium.

153. *If three forces, acting in one plane upon a* RIGID BODY, *keep it in equilibrium, their action lines either all meet at a point, or are all parallel.*

The lines of action of two of the forces may meet in a point, and their resultant must pass through that point and may replace the forces; but, since there is equilibrium, this resultant must be equal and opposite to the third force; hence, the line of action of the third force must pass through the intersection of the lines of action of the other two. If they do not meet they are parallel.

154. The polygon of pressures and the **parallelopipedon of pressures** follow directly from the *triangle of pressures*, giving expressions similar to the corresponding proportions for velocity, as in Articles 16 and 17.

EXAMPLES.

1. If the angle between two forces is right, what is the value of their resultant? If it is 0°? If 180°?
2. If the forces are 3 lbs., 4 lbs. and 5 lbs. respectively, and are in equilibrium, required the angle between the forces 3 and 4.
3. What is the angle between the forces when $P = F = R$?
4. If $P = F = 100$ lbs., and $\theta = 60°$, find R.
5. If $R = P + F$, required θ.
6. What will θ be when $-R = P - F$?
7. If $P = 50$ lbs., the angle $(P, F) = 35°$ and $(P, R) = 115°$, what will be the angle (F, R), and the values of the forces F and R for equilibrium?
8. A string 7 feet long has its ends fastened at two points in a horizontal line 5 feet apart; a weight of 20 lbs. is suspended at a point 3 feet from one end; required the tension on the two parts of the string.

9. Two forces represented by two chords of a semicircle passing from a point, show that their resultant will be represented by that diameter of the circle which passes through the point.

EXERCISES.

1. In Fig. 39, the diagonal joining C and B represents the resultant of what forces?
2. In the same figure what will represent the resultant of R and F?
3. In the same figure, if AD represents a force acting away from A, and another equal force should act along the line BG in the opposite direction, would they be in equilibrium?
4. Under what conditions will *two* forces be in equilibrium?
5. Can a particle be kept at rest by three forces whose magnitudes are as 4, 5, and 9? Or as 3, 4, and 8?
6. If R is the resultant of P and F, will P and F act when R is acting?

Resolution of Forces.

155. The resolution of forces consists in finding two or more components whose united action will equal that of the given force.

156. Rectangular Components.—Let R be a force whose components F and P form a right angle with each other. Let

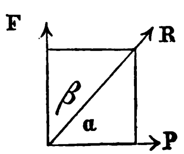


FIG. 41.

$\alpha =$ the angle between R and P,
$\beta =$ the angle between R and F;

then we have

$$P = R\cos\alpha;$$
$$F = R\cos\beta = R\sin\alpha;$$
$$R^2 = P^2 + F^2;$$

the last of which may be found by squaring and adding the two former; or, by observing that R is represented by the hypothenuse of a right-angled triangle, of which the sides represent P and F.

157. To find the angles a and β.—Since forces may act at all possible angles, and be applied at points anywhere in space, it is desirable to have a definite rule for determining the angles which they make with the axes of x and y.

In the first place, draw a line *away from* the origin of

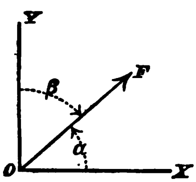


FIG. 42.

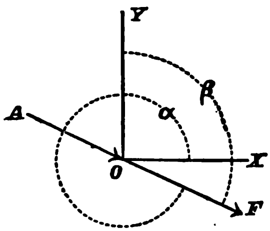


FIG. 43.

coördinates O, *parallel* to the line of action of the force, and *in the direction of action* of the force. If the force appears to act towards the origin as AO, Fig. 43, it must be prolonged so that it may be represented by OF.

Then, *secondly*, conceive that the angle a is generated by a line starting from the axis OX, and revolving about O *to the left*, until it coincides with the action-line of the force; the angle thus generated will be $+a$. In a similar way, $+\beta$ will be generated by a line revolving from the axis OY about O, *to the right*. Thus, in Fig. 42, a is an acute angle, in Fig. 43, it is nearly 360°, and in Fig. 44 it is between 90° and 180°; and β in Fig. 42 is acute, in Fig. 43, between 90° and 180°, and in Fig. 44, between 270° and 360°. It is, however, often more convenient to measure the angle

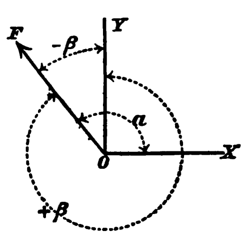


FIG. 44.

negatively; thus, in Fig. 43, $-\alpha$ is the angle XOF, and, in Fig. 44, $-\beta$ is YOF.

These rules are arbitrary, but a rigid observance of them secures uniformity in practice.

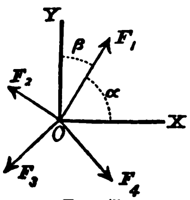


FIG. 45.

158. Problem.—*To find the rectangular components of any number of concurrent forces in a plane.*

Let O be the point of concurrent action, and through this point draw two lines, OX and OY, at right angles with each other. These lines, in Analytical Geometry, are called *rectangular axes.*

Let F_1, F_2, F_3, etc., represent the intensities of the respective forces;

α_1, α_2, α_3, etc., the angles which the forces make respectively with the axis of x;

β_1, β_2, β_3, etc., the corresponding angle with the axis of y;

X, the sum of the components along the axis of x, and

Y, the sum of the components along y;

then, according to Article 156, we readily find

$$X = F_1 \cos \alpha_1 + F_2 \cos \alpha_2 + F_3 \cos \alpha_3 + \text{etc.} = \Sigma F \cos \alpha;$$
$$Y = F_1 \cos \beta_1 + F_2 \cos \beta_2 + F_3 \cos \beta_3 + \text{etc.} = \Sigma F \cos \beta;$$

in which the expression $\Sigma F \cos \alpha$ means the sum of a series of terms of the form $F \cos \alpha$.

It is not necessary that the origin of coördinates be at the particle on which the forces act.

159. Resultant *of any number of concurrent forces.*

Let R be the resultant, then, according to Article 156, we have

$$R^2 = X^2 + Y^2.$$

If the given forces are in equilibrium among themselves we have

$$R = 0 \therefore X = 0 \text{ and } Y = 0.$$

160. To find the direction of the resultant we readily deduce from Fig. 41,

$$\cos(R, X) = \frac{X}{R}; \quad \cos(R, Y) = \frac{Y}{R}.$$

EXAMPLES.

1. Find the resultant of the concurrent forces in the plane xy; —$F_1 = 20$, $\alpha_1 = 30°$; $F_2 = 30$, $\alpha_2 = 90°$; $F_3 = 40$, $\alpha_3 = 150°$; and $F_4 = 50$, $\alpha_4 = 180°$, and find the angle between R and x.
2. If the forces $F_1 = 20$, $\alpha_1 = 180°$; $F_2 = 10$, $\alpha_1 = 270°$, are concurrent, find R.
3. If four forces are all equal to each other and concurrent, and $\alpha_1 = 0$, $\alpha_2 = 90°$, $\alpha_3 = 225°$, $\alpha_4 = 270°$, find R and the angle which it makes with the axis of x.

161. If the forces are referred to three rectangular axes, we have

$$X = F_1 \cos\alpha_1 + F_2 \cos\alpha_2 + \text{etc.} = \Sigma F \cos\alpha;$$
$$Y = F_1 \cos\beta_1 + F_2 \cos\beta_2 + \text{etc.} = \Sigma F \cos\beta;$$
$$Z = F_1 \cos\gamma_1 + F_2 \cos\gamma_2 + \text{etc.} = \Sigma F \cos\gamma;$$
$$R^2 = X^2 + Y^2 + Z^2;$$
$$\cos\alpha = \frac{X}{R}; \quad \cos\beta = \frac{Y}{R}; \quad \cos\gamma = \frac{Z}{R}.$$

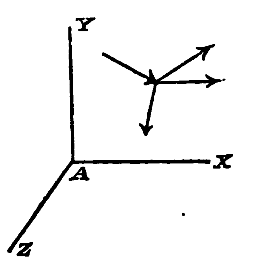


FIG. 46.

CHAPTER VII.

MOMENTS OF FORCES.

162. Definition.—The moment of a force is a *measure* of its effect in producing rotation, or of its tendency to produce rotation.

163. Measure of the Moment.—*The moment of a force, in reference to a point, is the product of the force into the perpendicular distance of the action-line of the force from the point.*

Let a particle w or w', Fig. 47, be connected with a

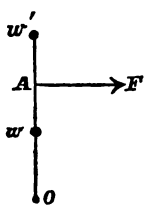


FIG. 47.

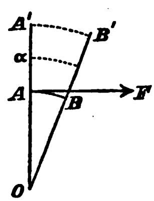


FIG. 48.

point O by a line without weight, and let a force F act at any point A of this line and perpendicularly to it. The effect of the force will vary directly as the distance A from the point O. Suppose that rotation has been produced, as shown in Fig. 48, the particle having been moved through an angle AOB. The point of application of the force will have moved over the space AB, and the work done by the force will be (Art. 92)

$$F.AB.$$

If the point of application were at A', it would have

moved over the space $A'B'$ in moving the particle w from A to B, and the work done by F would be

$$F \,.\, A'B'.$$

But,

$$AB : A'B' :: OA : OA';$$

hence,

$$F.AB : F \,.\, A'B' :: OA : OA';$$

that is, the effect of a force, in producing rotation, varies directly as the perpendicular distance of the force from the point about which rotation takes place. The effect evidently varies directly as the intensity of the force; hence, generally, the effect will vary as the product of the force into the distance of the action-line of the force from the point. This is called the moment, as given above.

164. *If the line of action of the force is inclined to the line OA*, resolve the force into two components, one F_2, acting along the line AO, the other, F_1, acting perpendicular to OA at the point of application A, of the given force. The former does not tend to produce rotation about O, but the latter acts in the same manner as in Fig. 47. Hence, we have, for the measure of the moment in this case,

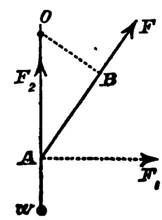


FIG. 49.

$$F_1 . OA = F \sin OAB . OA = F . OA \sin OAB = F . OB.$$

But OB is the perpendicular from O upon the line AB of the force F.

165. Axis of Moments.—Rotation about a point is always equivalent to a rotation about an axis which passes through the point, and perpendicular to the plane of motion of the particle. This line is called the axis of rotation, and, in reference to the moments of forces, is the *axis of moments*, or *moment axis*.

166. If the axis of rotation is *fixed*, and the line of action of the force is inclined to the plane of motion; in order to find the moment of the force, the force is resolved into two components, one of which is perpendicular to the plane of motion and the other parallel to it. The former has no moment in reference to the fixed axis, and the moment of the latter will be found by Article 164.

167. The point or axis about which the particle or body rotates may not only move in space, but may also change its position within a body.

168. Definitions.—The point O, where the axis of rotation pierces the plane of motion of the particle, or the plane in which it tends to move, is called *the origin of moments.*

The perpendicular OB, let fall from the origin of moments upon the action-line of the force, is called *the arm of the force.*

The moment of a force in reference to a point is the product of the force into its arm. If a is the arm of the force the moment will be

$$Fa.$$

The moment of the velocity is the product of the velocity into the perpendicular from the origin of moments upon the direction of motion. The direction of motion will be in a tangent to the path at the point where the velocity is considered. Let p be the perpendicular, then the moment of the velocity will be

$$pv.$$

The moment of the momentum is the continued product of the mass, velocity, and perpendicular from the origin of

moments upon the direction of motion. Let Q be the momentum, then

$$Qp = Mvp.$$

This principle is important in the solution of certain problems involving an aggregation of particles.

The moment of a force oblique to the axis of rotation is the product of the component of the force on a plane perpendicular to the axis into the arm of the component.

Generally, the moment of a *mechanical agent* is a measure of its importance in producing, or tending to produce, rotation.

169. The moment of a force may be represented by twice the area of a triangle, of which the base represents the magnitude and position of the force, and whose apex is at the origin of moments. For, the altitude of the triangle will be the perpendicular upon the force, and hence, will be the arm of the force.

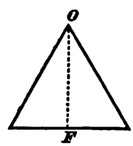


FIG. 50.

170. Sign of the Moment.—If a force tends to turn a system one way, it may be considered positive; then, if in the opposite direction, it will be negative. Either direction may be considered positive, but when chosen it must not be changed in the solution of a problem.

If a watch be placed at the origin of moments, with its face in the plane of the force, the moments of those forces which tend to turn the particle or body in the direction of the movement of the hands of the watch will be right-handed, and in the opposite direction left-handed.

171. The value of a moment *may be represented in magnitude and direction by the axis.* When the moment is positive, let the axis be represented *above* the plane, as in Fig. 51, and a distance laid off on it to *represent* its

magnitude. When the moment is negative, lay off the axis below the plane of the force, as in Fig. 52.

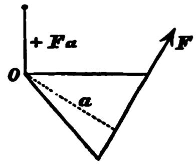


Fig. 51.

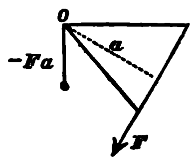


Fig. 52.

172. Composition of Moments.—Since moments are fully represented by lines, they may be compounded or resolved in the same manner as forces. Thus, if the forces are all in the same plane, their axes will be parallel, and, if they have the same origin, their axes will coincide; in which case, the resultant moment will be the algebraic sum of the several moments.

If O_r be the resultant moment, we have

$$O_r = Fa + F_1a_1 + F_2a_2 + \text{etc.}$$

If the forces act in different planes let all their *moment axes* pass through A, Fig. 53, and let

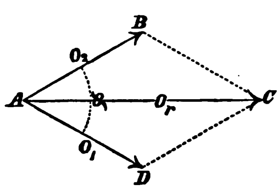


Fig. 53.

$O_1 = F_1a_1 =$ the moment of one force,

$O_2 = F_2a_2 =$ the moment of another force,

$\theta =$ the angle between the moment axes, and

$O_r =$ the resultant moment of O_1 and O_2;

then, as in Article 147, we have

$$O_r^2 = O_1^2 + O_2^2 + 2\,O_1O_2 \cos\theta.$$

If the axes do not pass through a common point, it may

will be proved that the resultant moment may be found from the preceding equation.

173. Proposition.—*If any number of concurring forces are in equilibrium, the algebraic sum of their moments will be zero.*

Let F_1, F_2, F_3, etc., be the forces acting upon a particle at A. Take the origin of moments at any point O. Draw OA, and let fall the perpendiculars Oa, Ob, Oc, etc., upon the action-lines of the forces; and let

$Oa = a_1$; $Ob = a_2$; $Oc = a_3$, etc.

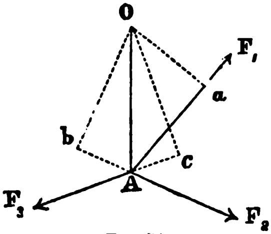


FIG. 54.

Since they are in equilibrium, the sum of the components of all the forces perpendicular to OA (see Article 142) will be zero; hence,

$$F_1 \sin OAF_1 + F_2 \sin OAF_2 + F_3 \sin OAF_3 + \text{etc.} = 0;$$

or

$$F_1 \frac{Oa}{OA} + F_2 \frac{Ob}{OA} + F_3 \frac{Oc}{OA} + \text{etc.} = 0.$$

Multiplying by OA, we have

$$F_1 . Oa + F_2 . Ob + F_3 . Oc + \text{etc.} = 0;$$

or

$$F_1 a_1 + F_2 a_2 + F_3 a_3 + \text{etc.} = \Sigma Fa = 0.$$

When there is equilibrium in reference to rotation, any one of the moments may be considered as equal in value but directly opposed to the resultant of the moments of all the other forces.

174. Unit of Moments.—If the force be given in pounds, and the arm in feet, the unit will be a foot-pound, and the moment will be a certain number of foot-pounds

of rotary effort. It will be observed that this is not the same as foot-pounds of work.

175. The origin of moments ***may be taken anywhere in the plane of the forces.*** This is evident from the preceding article; for, the point O, in Fig. 54, was not only chosen arbitrarily, but no trace of its position remains in the result. This is also evident from the fact that if the forces are in equilibrium their tendency to turn the body about *any point is zero.* In statical problems, therefore, the origin may be chosen arbitrarily.

176. The arm of a force in terms of Rectangular Coördinates.—Let F be a force, of which Aa is its line of action. Take the origin of coördinates at O, which is also taken as the origin of moments; then Oa will be the arm of the force.

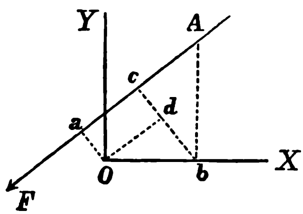


FIG. 55.

Take any point A, in the line of the force, and drop the perpendicular Ab, and from its foot drop the perpendicular bc upon aA, and draw Od parallel to aA.

Let $\alpha = dOb$; $\beta =$ the angle between F and $Y = cAb$.
$y = Ab$; $x = Ob$.

Then

$$cb = y \cos \alpha; \; db = x \cos \beta;$$

$$\therefore aO = cb - db = y \cos \alpha - x \cos \beta.$$

177. The origin of coördinates may be at one place and the origin of moments at any other, for the positions of both are arbitrary. Thus, in Fig. 55, the origin of moments may be taken at d, or b, or any other place, the origin of coördinates remaining at O.

178. Moments of Parallel Forces.—If the forces are parallel their *arms* will coincide. Thus, in Fig. 56, if O be the origin of moments, then will the arms of the forces F_1, F_2, F_3, etc., be Oa, Ob, Oc, etc., respectively.

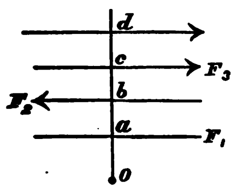


FIG. 56.

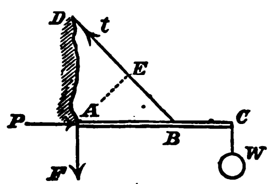


FIG. 57.

179. Problems.—1. *A weight W is suspended from one end of a horizontal bar AC, the other end of which rests against a vertical wall, the bar being held in position by a cord DB; required the tension of the cord.*

Take the origin of moments at A, where the bar touches the wall. The perpendicular AE upon the cord BD will be the arm of the tensile force of the cord, and AC will be the arm of W. Let t be the tension of the cord, then will $t.AE$ be the moment of the tension, and we have

$$t.AE = W.AC.$$

$$\therefore t = \frac{AC}{AE} W.$$

This problem illustrates the mechanical arrangement of the bones and muscles at the elbow-joint, by means of which the arm may be held in a horizontal position and support a weight in the hand. The joint is at A, the hand at C, and the muscle at DB; but in the arm the distance AE is much less in proportion to AC than that shown in the figure.

2. *A weight W is suspended by a cord DB, and pushed*

from a vertical by a bar AB; required the tension of the cord.

Take the origin of moments at A, and drop the perpendiculars AC and BE. The perpendicular BE will be of the same length as if it were drawn horizontally from A to a vertical through B. If t be the tension, we have

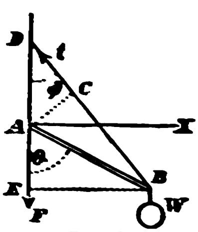


FIG. 58.

$$t \,.\, AC = W \,.\, BE.$$

$$\therefore t = \frac{BE}{AC} W.$$

If the length and inclination of AB be given, we have

$$BE = AB \sin \theta.$$

If the length of DB be also known, the angle $ADB = \phi$ may be found. Or, if the three lines AD, AB, DB, are given, the angles ϕ, θ, ABD, may be found by solving the triangle.

180. Choice of the Origin of Moments.—When the forces are in one plane, the problem may be solved by a single equation of moments, provided the origin of moments can be so taken that the moment of one force only will be unknown. It will be observed that the forces at the origin of moments have no moments, and hence do not enter the equation of moments. When there are several unknown forces, it is possible, in many cases, to find the value of all of them by taking the origin of moments at different places, so as to involve only one unknown quantity at a time.

181. Problems.—1. *In Fig. 58, suppose that the bar AB rests against a smooth surface DA, and is prevented from*

sliding upward by a string AE; required the tension of the string and the pressure against the surface.

Let

$F =$ the tension on AE, and

$X =$ the pressure against the surface.

If the origin of moments be taken at B, both the unknown forces F and X will enter the equation, and hence, neither will be determined. If it be taken at D the moment of F will be zero. The forces F and X are the components of the compression along BA; hence, the latter will not be considered when the former are. Taking the origin of moments at D, we have

$$X.DA = W.EB,$$

$$\therefore X = \frac{EB}{DA} W.$$

Now, taking the origin of moments at B, there results

$$F.BE = X.AE;$$

in which, substitute the value of X from above and reduce, and find

$$F = \frac{AE}{DA} W.$$

The compression along AB will be

$$\sqrt{X^2 + F^2} = \frac{\sqrt{EB^2 + AE^2}}{DA} W = \frac{AB}{DA} W.$$

The compression may be found directly by taking the origin of moments at D. Drop a perpendicular from D upon BA prolonged (which the student can draw in the

figure), and call it a. Let c = the compression; then the equation of moments will be

$$c \cdot a = W \cdot EB;$$

but

$$a = DA \sin\theta = DA \frac{EB}{AB}.$$

$$\therefore c = \frac{EB}{a} W = \frac{AB}{DA} W,$$

as before.

2. *Two forces, P and F, act as a pull upon the ends of a bar AB, both of which are inclined to the bar at known angles, the point C being fixed, and the distance CB known; required the distance AC for equilibrium.*

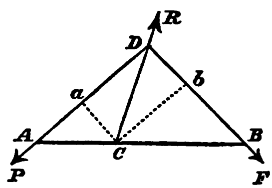


FIG. 59.

Take the origin of moments at C, and drop the perpendiculars Ca and Cb; then the equality of moments gives the equation

$$P.Ca = F.Cb.$$

The distance Cb is known from the equation

$$Cb = CB \sin CBb;$$

and AC from the equation

$$AC = Ca \operatorname{cosec} aAC.$$

Substitute in this equation the value of Ca found from the first equation, and Cb from the second, and we find

$$AC = \frac{F}{P} \frac{\sin CBb}{\sin aAC} CB.$$

The resultant of the forces P and F will pass through

their intersection D, and also through the fixed point C. The moment of the resultant, when the origin is at any point upon it, will be zero. Its value, however, may be found by taking the origin of moments at A or B.

3. *A weight P_1 is made to act vertically upward at the end of a rigid bar OA, and P_2 vertically downward at B, on the same bar, the bar being free to turn about the end O; required the distance Ob for equilibrium.*

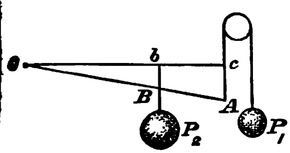


FIG. 60.

Take the origin of moments at O, and draw the horizontal line Oc, cutting the verticals through B and A at b and c respectively; then will the equation of moments give

$$P_1 . Oc - P_2 . Ob = 0.$$

$$\therefore Ob = \frac{P_1}{P_2} Oc.$$

If the distance bc between the weights is given, then

$$Oc = Ob + bc;$$

which, in the preceding equation, gives

$$Ob = \frac{P_1}{P_2 - P_1} bc.$$

EXAMPLES.

1. In Fig. 57, if $W = 20$ lbs., $AC = 2$ ft., $AB = 6$ in., and $AD = 4$ in., find the tension on the cord DB.
2. In the preceding example find the horizontal pressure against the wall.
3. In Fig. 57, if the surface at A is perfectly smooth, find the vertical force F applied at A, which will just prevent sliding.

4. If the cord DB, Fig. 57, is inclined 45°, what must be the distance from A to B that the tension on the string shall equal the weight W?

5. In Fig. 58, if $DB = 2AB$, $\theta = 45°$, and $W = 50$ lbs., required the tension on the cord and the compression on the bar.

6. A strut BC, free to turn about its lower end, supports a weight W from its upper end, the strut being held by a cord AC, one end of which is attached to the upper end of the strut and the other to a point A in the horizontal plane; required the tension of the cord.

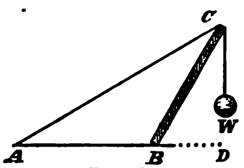


FIG. 61.

7. In the preceding example, what condition must be fulfilled that the tension of the cord will equal the weight W?

8. In Fig. 61, if $W = 500$ lbs., $AB = 6$ ft., $BD = 4$ ft., and $DC = 8$ ft., what will be the compression upon CB?

9. In Fig. 60, if $P_1 = P_2$ and $bc = 2$ ft., what will be the distance Ob for equilibrium?

10. In Fig. 60, if $Ob = 2$ ft., and $P_1 = P_2$, what will be the distance bc for equilibrium?

Couples.

182. Definition.—*Two equal parallel forces, acting in contrary directions and whose lines of action are not coincident, constitute a couple.*

The last equation of the preceding article is

$$Ob = \frac{P_1}{P_2 - P_1} bc;$$

in which, if $P_2 = P_1$, it becomes

$$Ob = \infty;$$

that is, in order that two such forces shall be in equilibrium, in reference to rotation, the origin of moments must be at an infinite distance from the forces; in other words, *two equal parallel forces, whose directions are contrary and lines of action not coincident, cannot be in equilibrium in reference to rotation.* Such a system has received a special name, called *a couple.*

183. The office of a couple *is to produce, or to tend to produce, rotation only.* For, the only other effect which it can produce is that of translation; but, since the forces are equal and directions contrary, whatever translation is produced by one force in any direction will be exactly neutralized by the other in the opposite direction; hence, they cannot produce translation.

184. Moment of a Couple.—In Fig. 62, let O be the origin of moments, and P at A equal P at B, one moment being right-handed and the other left-handed. The moment will be

$$P.OB - P.OA \text{ (not equal 0).}$$

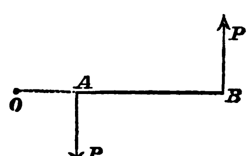


FIG. 62.

But,

$$OB = OA + AB;$$

hence, the preceding expression becomes

$$P.OA + P.AB - P.OA;$$

or, simply

$$P.AB;$$

that is, the moment of a couple is *the product of one of the forces into the perpendicular distance between the lines of action of the forces.*

This is independent of the origin of moments. If the

origin be at A, upon the line of action of one of the forces, the moment of the couple will be the same as the moment of one of the forces.

185. A couple can be equilibrated only by an equivalent couple having a contrary moment. For, the only effect being rotation, such a system of forces must be employed as will produce a contrary rotation, and this requires an equivalent couple.

186. A resultant couple is one which will produce the same effect as the several couples.

If P_1, P_2, etc., be the forces of several couples all in one plane,

a_1, a_2, etc., be their respective arms,

R the force of a resultant couple, and

r the arm of the resultant couple;

then

$$Rr = P_1a_1 + P_2a_2 + P_3a_3 + \text{etc.}$$

If the couples are in equilibrium, any one of them may be taken as equal and contrary to the resultant of all the others.

187. Proposition. *If two couples, having equal moments, but whose directions of action are contrary, act upon a body, they will equilibrate each other.*

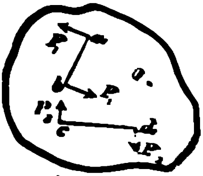


FIG. 65.

This is evident from Article 185, but the proposition is presented here in order to show that the forces constituting the couples may be applied at any point of the body, and that the arms of the couples need not be parallel.

Conceive that any point O in the body is fixed, and taken as the origin of moments, then will the moments of the couples in reference to this point be

$$P_1 . ab \text{ and } P_2 . cd;$$

But, since their moments are assumed to equal each other, we have

$$P_1 . ab - P_2 . cd = 0;$$

hence, there is no tendency to turn about O. The same may be shown in reference to any other point of the body; hence, the body will be in equilibrium in reference to rotation.

188. Proposition.—*A force, acting at any point of a body, is equivalent to an equal parallel force at the origin of moments, and a couple whose moment is the moment of the original force.*

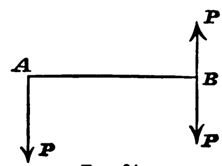


FIG. 64.

Let the force P be applied at A, and the origin of moments be at B. At B introduce two equal and opposite forces, each equal and parallel to the original force P. Since the two forces at B neutralize each other, the effect of the three forces will be the same as that of the single force P. But, by making a new combination of the forces, we have the force acting down at A, combined with the equal parallel force at B acting upward, constituting a couple whose arm is AB, and the force P at B acting downward.

When a body is free to move, a single force acting upon it *may* produce both rotation and translation, and it may be shown that it will produce both, unless the line of action of the force passes through the centre of the mass of the body.

189. Proposition.—When several forces have a resultant, the sum of their moments will be zero when the origin of moments is upon the line of the resultant, for the moment of the resultant will be zero.

Three Parallel Forces.

190. The relation between three parallel forces in eq librium may be found by means of the principles of n ments, and the result may be extended to any number parallel forces.

In Fig. 65, let the forces R, F, P, act in parallel lir and be in equilibrium.

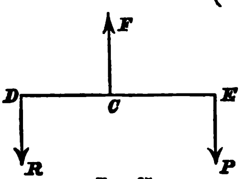


FIG. 65.

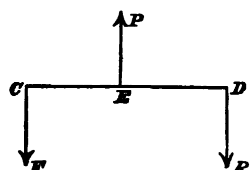


FIG. 66.

Taking the origin of moments at D, we have

$$F.DC = P.DE;$$

and if the origin of moments be at E, we have

$$F.CE = R.DE.$$

Adding these equations gives

$$F(DC + CE) = (P + R)DE;$$

but,

$$DC + CE = DE,$$

which, in the preceding equation, gives

$$F = P + R;$$

hence, the force F, acting in one direction, equals the su of the two forces acting in the contrary direction.

If F and P are given, we have, by transposition,

$$F - P = R.$$

The same principles apply to Fig. 66, in which t directions of P and F are the reverse of those in Fig. 6

By transposing R in the last equation we have

$$F - P - R = 0;$$

in which, if F be called the typical force, and the algebraic signs be understood, we may write it

$$\Sigma F = 0;$$

and this expression is true for any number of parallel forces.

191. A rigid body, being acted upon by any number of parallel forces in one plane, it is necessary and sufficient for equilibrium that we have

$$\Sigma Fa = 0;$$

$$\Sigma F = 0;$$

the former of which will determine equilibrium in reference to rotation, and the latter in reference to translation.

192. *A single force and a single couple in one plane are equivalent to a single force equal and parallel to the original single force, but having another point of application.*

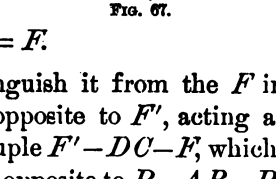


FIG. 67.

If they are parallel, as in Fig. 67, the resultant of the forces in reference to translation will be

$$F + P - P = F.$$

Call this resultant F', to distinguish it from the F in the figure. A force equal and opposite to F', acting at some point D, will produce the couple $F' - DC - F$, which, for equilibrium, must be equal and opposite to $P - AB - P$. Hence, to find DC, we have

$$F.DC = P.AB$$

$$\therefore DC = \frac{P}{F} AB.$$

If the forces are not parallel, combine F with P, and their resultant with $-P$, and the same result will be obtained.

193. Remark.—The principle of moments, strictly speaking, is applicable only to problems involving extended masses; for, although we speak of the moment of a force in producing the rotation of a particle, yet, in order to realize it, it is necessary to assume that the particle is connected with the point about which rotation takes place, by means of a rigid bar, which is itself a finite body and not a particle.

194. Proposition.—*A system of forces acting in one plane, if not in equilibrium, must be equivalent to a single force or to a couple.*

For, the resultant of two forces may be found, and the resultant of that resultant and a third force, and so on, and if their lines of action thus intersect each other a single resultant may be found; otherwise a single couple may be found.

195. Proposition.—*A system of forces in one plane, acting on a rigid body will be in equilibrium if the algebraic sum of the moments of the forces vanishes in reference to three points in the plane not in a straight line.*

For, if they are not in equilibrium in reference to rotation, the algebraic sum of the moments could not vanish for any point in the plane; and if they had a single resultant, then the moments would vanish only for points on the line of the resultant.

196. Problems.—1. *A prism AF is acted upon by a couple in the plane of the upper base; required the value of the couple in the plane of the lower base that will equilibrate the former.*

The forces in the plane of the upper base tend to turn

he prism about an axis perpendicular to the base; similarly the couple in the lower base will tend to turn it about the same axis; hence, if their moments are equal and contrary, they will equilibrate each other.

This shows that couples of equal moments in parallel planes are equivalent.

The couples, in this case, twist the body upon which they

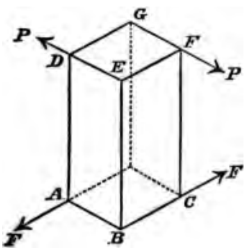


FIG. 68.

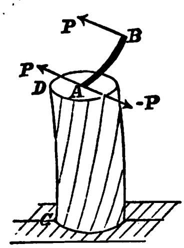


FIG. 69.

act. This effect is called torsion. The amount of torsion depends upon the properties of the material, as well as upon the size of the body and the moment of the couple. The properties of materials are investigated in works upon the *Resistance of Materials*.

A *single force* applied at the end of a lever, as in Fig. 69, will not only twist the body, but will push it sidewise. For, as we have seen in Article 188, it will be equivalent to a couple whose moment is

$$P.AB,$$

and a force

$$P,$$

applied at A, the former of which twists the body and the latter pushes it sidewise. This may be easily illustrated by the student in a variety of ways, such as turning an auger by one handle, twisting a long rod by means of a single-handed wrench, etc.

2. *A door, gate, or frame, supported by two hinges, carries a weight; required the pressure upon the hinges.*

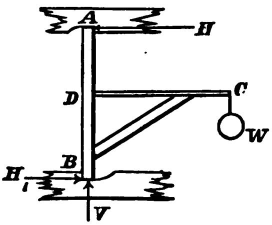


FIG. 70.

When hinges are employed, they may be so set that one o the other will carry all the vertical pressures. In Fig. 70, there being no provision for carrying any of the vertical pressures at the upper bearing, they will all be supported at the lower end.

Taking the origin of moments at B, we have

$$H.BA = W.DC$$

$$\therefore H = \frac{DC}{BA}W.$$

Similarly, taking the origin at A, we have

$$H_1.BA = W.DC$$

$$\therefore H_1 = \frac{DC}{BA}W;$$

hence,

$$H = H_1;$$

and as they are parallel they constitute a couple. The only remaining force is the vertical one at B, and is called V. This force, combined with W, must constitute the equilibrating couple; hence,

$$V = W.$$

The total pressure at the lower end is the resultant of V and H_1, and hence is

$$\sqrt{\frac{DC^2}{BA^2}W^2 + W^2} = \frac{\sqrt{DC^2 + BA^2}}{BA}W.$$

EXAMPLES.

If three forces are represented in magnitude, *position*, and direction of action by the sides of a triangle taken in their order, show that they are equivalent to a couple.

In Fig. 71, if the forces act along the sides of a triangle, P from B toward C, F from C toward A, and R from B toward A; show that for equilibrium in reference to rotation, $R = P + F$.

Equal weights are suspended at the corners of a triangle; required the point where the triangle must be supported that there will be equilibrium.

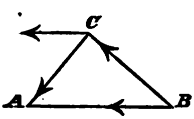


FIG. 71.

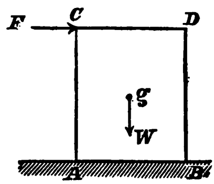


FIG. 72.

Two men carry 175 lbs. between them on a pole, resting on one shoulder of each; the weight is twice as far from one as from the other; how much weight does each carry, neglecting the weight of the pole?

If a prismatic block of stone, whose width AB is 2 ft., height AC is 3 ft., weighs 500 lbs., and it be considered that the whole weight acts at the centre g, what force, acting horizontally at C, will just turn the block about the edge B?

A man, whose weight is 175 lbs., desires to raise a body which weighs 4,000 lbs. by means of a lever 8 feet long; one end of the lever being placed under the

body, how far from the end shall the fulcrum b
placed so that his weight at the other end shall jus
balance the body?

EXERCISES.

1. The force being given in pounds, and the arm in feet, is it proper t say—a moment of a certain number of foot-pounds? Foo pounds of what?
2. What is the meaning of foot-pounds of work?
3. What is meant by foot-pounds of momentum per second?
4. If velocity is given in feet, and the arm also in feet, what will be th unit of the moment of momentum?
5. Can a single force acting upon a rigid body produce rotation if ther is not a fixed point in the body?
6. Will two couples acting upon a rigid body in planes at right angl with each other produce translation?
7. If a person supports a weight of 100 lbs. suspended from a rod upo his shoulder, and he pulls down upon the rod with a force of 25 lb with his hands so as to balance the weight, how much more tha his own weight will be the pressure of his feet upon the earth.?
8. If a hole be bored by an auger, is the resistance to the cutting equivalent to a couple?
9. In Fig. 70, will the pressure H at A be increased if the weight W be placed at a greater distance from D?

CHAPTER VIII.

PARALLEL FORCES.

197. Parallel Forces are such as act along parallel lines. They may be conceived as concurring in a point at an infinite distance, but this would be equivalent to saying that they do not actually concur. They are forces not acting upon a single particle, but upon the several particles of a body.

198. The Resultant of Parallel Forces.—Let R be the resultant of the parallel forces F_1, F_2, etc.; then, according to Article 191, we have

$$R = F_1 + F_2 + F_3 + \text{etc.} = \Sigma F.$$

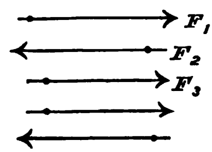


FIG. 73.

It will be observed that the value of the resultant is independent of the points of application of the forces. If the forces are in the plane xy, and are resolved parallel to the axes x and y, we have, according to Article 156,

$$X = (F_1 + F_2 + F_3 + \text{ etc.}) \cos a = \cos a \Sigma F;$$

$$Y = (F_1 + F_2 + F_3 + \text{etc.}) \cos \beta = \cos \beta \Sigma F = \sin a \Sigma F.$$

Squaring and adding, gives

$$R^2 = X^2 + Y^2 = (\Sigma F)^2 (\cos^2 a + \sin^2 a)$$
$$= (\Sigma F)^2$$
$$\therefore R = \Sigma F;$$

which is the same as given above.

If the given forces are in equilibrium, we have

$$R = 0;$$

$$\therefore \Sigma F = 0; \ X = 0; \ Y = 0.$$

199. The expression for the moments of parallel forces is given in Article 178, and the condition for equilibrium in reference to rotation, in Article 191. These expressions are independent of the points of application of the forces; but when these points are given the equation of moments is not only simplified, but it is found that there is always a point on the line of action of the resultant, called *the centre of parallel forces*, which possesses an important property.

200. Centre of Parallel Forces.—*The centre of parallel forces is that point through which the resultant will constantly pass as the forces are rotated about their points of application, the forces remaining constantly parallel as they are rotated.*

To illustrate, let the parallel forces P, F, and R be in equilibrium, having their points of application at P, A, and C in the straight line PC. The point C will be the point of application of the resultant of P and R. Draw CE perpendicular to the lines of action of the forces; then, since there is equilibrium, we have

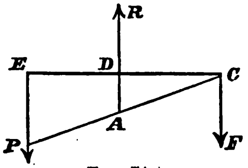


FIG. 74.

$$P.EC = R.DC. \quad . \quad . \quad . \quad (1)$$

Conceive that the forces are revolved through an angle of 90°, retaining their relative directions of action. The force R will then act at A, parallel to EC and to the right (or left), and P will act at the point P, also parallel

to EC and to the left (or right). In the new position the arm of R, in reference to C, will be DA, and of P, will be EP; hence, if there is equilibrium, we will have

$$P.EP = R.DA. \quad . \quad . \quad . \quad (2)$$

For, from the similar triangles CDA and CEP, we have

$$EC : DC :: EP : DA.$$

$$\therefore DC = \frac{DA}{EP} EC;$$

which, substituted in equation (1), gives

$$P.EP = R.DA;$$

which is the same as equation (2); hence, the resultant in the new position will pass through C.

In a similar way it may be shown that it will pass through C for any amount of rotation of the forces P and R; hence, C is the centre of the parallel forces P and R.

Similarly, A is the centre of the parallel forces F and P, if C and P are the points of application of the respective forces. But if E be the point of application of P, and C of F, then A will not be the centre of those forces.

The centre of two parallel forces will be on the line joining their points of application; and by combining their resultant with a third force in a similar way, the centre of three forces may be found, and so on for any number of forces.

201. To find the Centre of any Number of Parallel Forces.—Let the forces be in the plane xy, their points of application being at A, C, E, etc. The centre

of the forces F_1 and F_2 will be at B, where their resultant intersects the line AC. Let

x_1, y_1 be the coördinates of A,
x_2, y_2 " " C,
x_3, y_3 " " E,
. . . .
x', y' " " B,
x'', y'' " " D,
. . . .

and

$\bar{x}, \bar{y}$ be the coördinates of the centre of all the forces, which, being on the resultant, is called *the point of application of the resultant.*

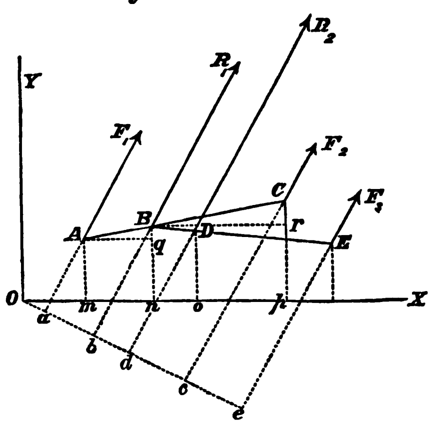


FIG. 75.

Draw Aq and Br parallel to OX, Am, Bn, etc., parallel to OY, and Oe perpendicular to the lines of action of the forces. The triangles AqB and BrC are similar, and give

$$AB : BC :: Aq : Br$$
$$:: On - Om : Op - On$$
$$:: x' - x_1 : x_2 - x'.$$

But, in reference to the point B, we have

$$F_1 . ab = F_2 . bc;$$

and because ac and AC are cut by parallel lines, we have

$$ab : bc :: AB : BC;$$

which, by means of the preceding proportion, gives

$$ab : bc :: x' - x_1 : x_2 - x';$$

and this, in the above equation, gives

$$F_1(x' - x_1) = F_2(x_2 - x');$$

or,

$$F_1x' - F_1x_1 = F_2x_2 - F_2x';$$

which, by transposing, gives

$$(F_1 + F_2)x' = F_1x_1 + F_2x_2;$$

$$\therefore R_1x' = F_1x_1 + F_2x_2.$$

In a similar way we would find

$$R_2x'' = R_1x' + F_3x_3$$
$$= F_1x_1 + F_2x_2 + F_3x_3;$$

and, finally,

$$R\bar{x} = F_1x_1 + F_2x_2 + \text{etc.} = \Sigma Fx;$$

and similarly,

$$R\bar{y} = F_1y_1 + F_2y_2 + \text{etc.} = \Sigma Fy.$$

From these equations, we have

$$\bar{x} = \frac{\Sigma Fx}{R};$$

$$\bar{y} = \frac{\Sigma Fy}{R}.$$

202. If the System be referred to Three Rectangular Axes, x, y, z, in which α, β, and γ are the angles which

the lines of action of the forces make with the respective axes; then the equations for equilibrium will be

$$X = \cos a \Sigma F = R \cos a;$$
$$Y = \cos \beta \Sigma F = R \cos \beta;$$
$$Z = \cos \gamma \Sigma F = R \cos \gamma;$$

$$R = \Sigma F;$$

$$R\bar{x} = \Sigma Fx; \quad R\bar{y} = \Sigma Fy; \quad R\bar{z} = \Sigma Fz;$$

The last three equations are the moments of the forces in reference to the respective coördinate planes.

If the given forces are in equilibrium in reference to translation, we have

$$R = 0;$$

and, if they are also in equilibrium in reference to rotation, we have

$$\Sigma Fx = 0; \quad \Sigma Fy = 0; \quad \Sigma Fz = 0$$

$$\therefore \bar{x} = \frac{0}{0}; \quad \bar{y} = \frac{0}{0}; \quad \bar{z} = \frac{0}{0};$$

hence, the centre of an equilibrated system is indeterminate.

203. Centre of a Mass.—*The centre of the mass which constitutes a body, is a point so situated that, if its distance from any axis be multiplied by the entire mass, the product will equal the sum of the products obtained by multiplying each elementary mass by its distance from the same axis.*

This point will be determined when its position in reference to three rectangular planes is known.

Let M = the total mass of the body;
m = an elementary mass;
x, y, z, the coördinates of any element, and
$\bar{x}$, $\bar{y}$, $\bar{z}$, the coördinates of the centre of the mass;
then, according to the definition, we have

$$M\bar{x} = \Sigma mx,$$
$$M\bar{y} = \Sigma my,$$
$$M\bar{z} = \Sigma mz.$$

These equations are applicable to several bodies. If the origin of coördinates be at the centre of the mass, we have

$$\bar{x} = 0; \quad \bar{y} = 0; \quad \bar{z} = 0$$
$$\therefore \Sigma mx = 0; \quad \Sigma my = 0; \quad \Sigma mz = 0.$$

EXAMPLES.

1 Two parallel forces whose magnitudes are 6 and 11, acting in the same direction upon a rigid line, have their points of application, A and B, 5 feet from each other; required the point of application of the resultant.

2. In the preceding example, find the point of application of the resultant if the forces act in contrary directions.

3. If the weights 2, 3, 4, and 5 lbs. act perpendicularly to a straight line at the respective distances of 2, 3, 4, and 5 feet from one extremity, what will be their resultant and its point of application?

4. Let the weights 3, 4, 5, and 6 act perpendicularly to a straight line at the points A, B, C, and D, the distances $AB = 3$ feet, $BC = 4$ feet, and $AD = 5$ feet; required the resultant, and the distance from A to the point of application E of the resultant.

5. If two parallel forces, P and F, act in contrary directions at the points A and B, and make an angle ϕ with the line AB; find the moment of each in reference to the point of application of the resultant.

EXERCISES.

1. Has a statical *couple* a centre of force?
2. Will several parallel forces always be in equilibrium if the sum of their moments is zero?
3. When may the resultant of parallel forces be zero, and the system not be in equilibrium?
4. If a system is in equilibrium why may the centre of force be at any point?
5. If the mass of a body is homogeneous, will the centre of the mass be at the geometrical centre of the body?
6. If, in a sphere, the density varies directly as the distance from the centre in all directions, will the centre of the mass be at the centre of the sphere?
7. State different laws according to which the density in a sphere may vary, and have the centre of the mass at the centre of the sphere.

CHAPTER IX.

CENTRE OF GRAVITY.

204. The lines of action of the force of gravity con- erge towards the centre of the earth; but the distance the centre of the earth from the bodies which we have casion to consider, compared with the size of those dies, is so great, that we may consider the lines of action the forces as parallel. The number of the forces of avity acting upon a body may be considered as equal to e number of particles composing the body.

205. The Centre of Gravity *of a body* may be defined the centre of the parallel forces of gravity acting upon e body; and hence the centre of gravity of bodies may found in the same way as the centre of parallel forces.

206. The Resultant of the Force of Gravity equals e weight of the body; or

$$R = W.$$

207. If a body be supported at its centre of gravity, and e body be turned about that point, it will remain in uilibrium in all positions, for it will be equivalent to rning the forces through the same angle.

208. Proposition.—*If a body be suspended at any oint, then, for equilibrium, the vertical through the cen- e of gravity will pass through the point of support.*

Let the body be suspended at the point c, the centre of ravity being at a. If the vertical through a does not ass through c, join the points c and a by the line ca, draw

the vertical cb, and the horizontal ab. The weight W may be represented by the line cb, of which the components are ba and ca. The component ba will cause the body to turn about c, causing the centre a to approach the vertical cb. If a be in the vertical cb, either above or below the support, there will be no horizontal component, and the body will be in equilibrium.

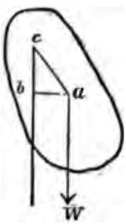


FIG. 76.

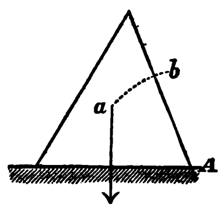


FIG. 77.

FIG. 78.

209. Stable Equilibrium.—In stable equilibrium, if the body be turned slightly from its position of rest, it will tend to return to its former position. Thus, the body represented in Fig. 77 is in stable equilibrium. It appears also from Figs. 76 and 77, that when the equilibrium is stable, and the body is turned about the support, the centre of gravity will be *raised*. In this case, therefore, the centre of gravity is the lowest possible. The measure of the stability is the amount which the centre of gravity is raised in overturning the body.

210. Unstable Equilibrium.—A body is in a condition of unstable equilibrium, if, when it is turned slightly from its position of rest, it departs, or tends to depart, farther from that position. This is illustrated by Fig. 78. In this case the centre of gravity will fall, when the body is turned about its support. The centre of gravity will be the highest possible.

211. Indifferent Equilibrium is that condition, in which a body will remain in equilibrium after being slightly disturbed. A sphere or cone resting on a horizontal plane is an example of indifferent, or *neutral*, equilibrium.

212. Trial Methods.—The preceding principles enable one to determine, *in an experimental manner*, the centre of gravity of bodies. Thus, in Fig. 79, let the body be carefully balanced upon a knife-edge, and, when balanced, the line of support be carefully marked upon the body. Then balance it in a similar way along another line; the intersection of the lines will be vertically under the centre of gravity, and, if the body be a thin plate, we say that the centre of gravity is at their intersection.

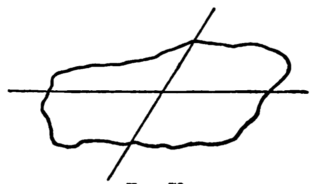


FIG. 79.

213. If a body be suspended at a point a, the centre of gravity will be vertically under it. Draw the vertical line ab upon (or within) the body; then suspend it at

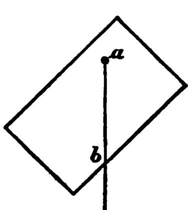


FIG. 80.

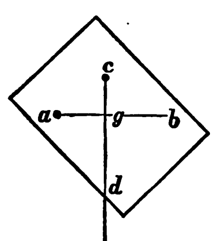


FIG. 81.

another point c, Fig. 81, and mark the vertical line cd. The centre of gravity will be at g, the intersection of the lines ab and cd.

EXERCISES.

1. If a carriage stands upon a side hill, what condition must be fulfilled in order that it shall not overturn? What must be the condition that it shall overturn?
2. A man stands upon a floor; how far can he lean forward or backward and not fall over?
3. When a man moves his head forward, what other motion must his body have that he may remain in equilibrium upon his feet?
4. Why will not a table be as stable when standing upon two legs as upon three?
5. Why is it more difficult to overturn a body like Fig. 77 than it is one like Fig. 78, the bodies being of equal weight?
6. If a book be suspended at one corner, why will its edges be inclined to a vertical?
7. May a body be in a state of neutral equilibrium in reference to a disturbance in one direction, and stable in reference to another?

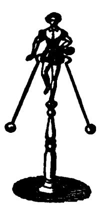


Fig. 82.

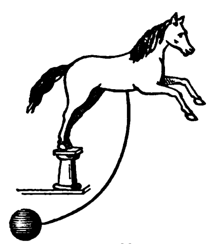


Fig. 83.

8. Explain how the toy shown in Fig. 82 may be in stable equilibrium.
9. Explain how the toy horse shown in Fig. 83 stands upon the post without falling off.

Centre of Gravity of Heavy Particles.

214. Centre of Gravity of two Particles.—Let P be the weight of a particle at A, and W, that at C. The centre of gravity will be at some point B, on the line joining A and C. Hence,

Fig. 84.

$$P . AB = W . BC;$$

but,

$$AB + BC = AC;$$

which, combined with the preceding equation, gives

$$AB = \frac{W}{W+P} AC.$$

If

$$P = W,$$

then

$$AB = \tfrac{1}{2} AC.$$

215. Centre of Gravity of several Heavy Particles.—Let w_1, w_2, w_3, etc., be the weights of the particles. Join w_1 and w_3 by a straight line and find their centre of gravity A, as in the preceding article. Join A with w_2 and find the centre of gravity B, which will be the centre of gravity of the three weights w_3, w_1, w_2. In a similar way find C, the centre of gravity of the four weights. In this way the centre of gravity of any number of weights

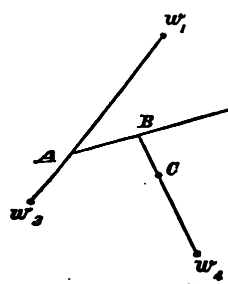


FIG. 85.

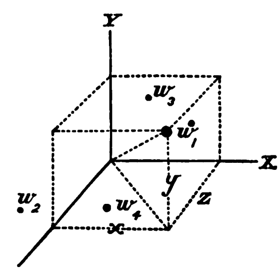


FIG. 86.

may be found. It is not necessary that the weights be in one plane; they may be distributed in any manner in space.

216. If the Positions of the Particles are referred to Three Rectangular Axes, let

w_1, w_2, w_3, etc., be the weights of the respective particles,

x_1, y_1, z_1 the coördinates of w_1 and similarly for w_2, w_3, etc.,

$\bar{x}$, $\bar{y}$, $\bar{z}$ the coördinates of the centre of gravity of all the weights, and

W the sum of all the weights;

then, we have

$$W = w_1 + w_2 + w_3 + \text{etc.} = \Sigma w;$$

and, according to Article 202, we have

$$\bar{x} = \frac{\Sigma wx}{W}; \quad \bar{y} = \frac{\Sigma wy}{W}; \quad \bar{z} = \frac{\Sigma wz}{W}.$$

EXAMPLES.

1. Two particles are joined by a straight line; if one is n times as heavy as the other, find the position of the centre of gravity.
2. If three equal particles are at the vertices of a triangle, find the position of the centre of gravity.
3. If the weights of three particles are as 1 to 2 to 3, and are placed at the vertices of an equilateral triangle, find the position of the centre of gravity.
4. If four equal weights are at the vertices of a triangular pyramid, find the position of the centre of gravity.

Centre of Gravity of Lines.

217. Straight Lines.—By a line, we here mean a material line, whose transverse section is very small, such as a very fine wire.

The centre of gravity of a uniform straight line is at its middle point.

For, we may conceive it to be composed of pairs of articles, each of which is at the same distance from the middle point; and as this point will be the common centre of gravity of all the pairs, it will be the centre of gravity of the line.

218. Centre of Gravity of the Perimeter of a Triangle.—Let ABC be the triangle. The centre of gravity of the sides will be at their middle points, D, E, F. Join these points. The weight at E will be to that at D as the length of CB is to the length of AC. Divide the line DE at the point G, so that

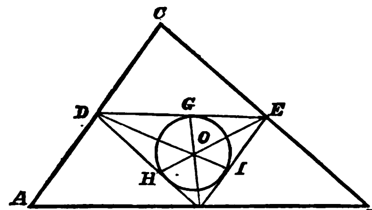


FIG. 87.

$$DG : GE :: BC : AC,$$

then the point G will be the centre of gravity of the two lines AC and CB. Similarly, I will be the centre of gravity of BC and AB, and H, that of AB and AC. The construction gives

$$DF = \tfrac{1}{2}BC; \; DE = \tfrac{1}{2}AB; \; FE = \tfrac{1}{2}AC;$$

hence, the preceding proportion gives

$$DG : GE :: \tfrac{1}{2}BC : \tfrac{1}{2}AC$$
$$:: DF : FE.$$

Therefore, the line drawn from F to G will bisect the angle F; and, similarly, for DI and EH.

The centre of gravity of the three sides of the triangle ABC, will be in the line GF; and, similarly, it will be in the lines DI and HE; hence, it will be at their intersection, which will be *the centre of the circle inscribed in the triangle DEF.*

219. Symmetrical Lines.—The centre of gravity of lines which are symmetrical in reference to a point, will be at that point. Thus:—

The centre of gravity of the circumference of a circle, or an ellipse, is at the geometrical centres of those figures:

The centre of gravity of the perimeter of an equilateral triangle, or of a regular polygon, is at the centre of the inscribed circle:

The centre of gravity of the perimeter of a square, rectangle or parallelogram, is at the intersection of the diagonals of those figures.

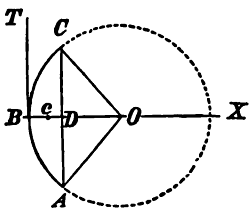


Fig. 88.

220. Centre of Gravity of a Circular Arc.—Let ABC be an arc of a circle, O its centre, B its middle point, and AC its chord. The centre of gravity will be on the radius BO, at some point c, such that (see Article 234),

$$\text{Arc } ABC : \text{radius } BO :: \text{chord } AC : Oc.$$

$$\therefore Oc = \frac{BO.AC}{ABC}.$$

EXAMPLES.

1. Find the position of the centre of gravity of the edges of a rectangular box.
2. Find the position of the centre of gravity of the edges of a regular pyramid having a square base, and whose altitude equals the length of one side of the base.
3. Find the centre of gravity of the semi-circumference of a circle.

Ans. $Oc = \frac{2r}{\pi}$.

4. In Fig. 88, if the angle $AOC = 60°$, find the position of the centre of gravity of the arc ABC.

Ans. $Oc = \frac{3r}{\pi}$.

5. Find the distance from the centre of a circle to the centre of gravity of a quarter of the circumference of the circle.

Ans. $Oc = 2\sqrt{2}\frac{r}{\pi}$.

Centre of Gravity of Surfaces.

221. Definition.—A surface here means a very thin plate or shell.

222. The Centre of Gravity of a Plane Triangle *is in the line joining the vertex with the middle point of the base, and at one-third the length of the line from the base.*

Let ABC be a triangle. Consider the triangle as composed of an indefinitely large number of straight lines parallel to the base AB. The centre of gravity of each of these lines is at their middle point; hence, the centre of all of them will be at some point in the line passing through their centres; which point will be the centre of gravity of the triangle. Let D be the middle point of AB; join C and D, then will the centre of gravity of the triangle be in the line CD. Similarly, it will be on the line AE, drawn from A to the middle point of BC; hence, it will be at g, the intersection of DC and AE. To find the distance Cg, draw DE, then will the similar triangles DEg and AgC give

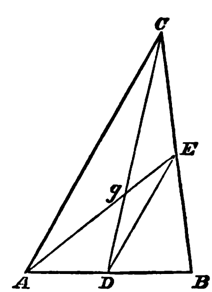


FIG. 89.

$$\frac{DE}{AC} = \frac{gD}{gC} = \tfrac{1}{2}$$

$$\therefore gC = 2gD.$$

Adding gD to both members, gives

$$gC + gD = DC = 3gD$$

$$\therefore gD = \tfrac{1}{3}DC.$$

Similarly,

$$gE = \tfrac{1}{3}AE.$$

We may readily find that, the perpendicular distance of the centre of gravity from the base, equals one-third of the altitude.

223. Symmetrical Figures.—The centre of gravity of the surface of a circle, or of an ellipse, is at the geometrical centre of the figure; of an equilateral triangle, or a regular polygon, it is at the centre of the inscribed circle; of a parallelogram, at the intersection of the diagonals; of the surface of a sphere, or an ellipsoid of revolution, at the geometrical centre of the body; of the convex surface of a right cylinder, at the middle point of the axis of the cylinder. The centre of gravity of the convex surface of a regular right pyramid, or of a right cone having a circle for its base, is on the axis of the figure at one-third the altitude from the base; for, the surface may be considered as composed of triangles having a common apex.

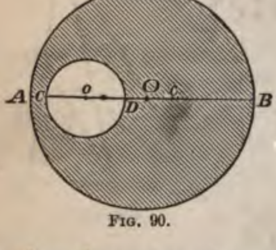


Fig. 90.

224. To find the Centre of Gravity of a Part of a Body, *when the centre of gravity of the whole and of the remaining part are known.*

Let AB be one area, CD the other, O the centre of

gravity of the former, o that of the latter, and c the centre of gravity of the part remaining after removing the area CD. The point c will be on the line through oO. Take A as the origin of moments. The weights are directly proportional to the areas, and the moment of the whole equals the sum of the moments of all the parts, hence

$$Area\ AB \times AO = Area\ CD \times Ao + (Area\ AB - Area\ CD) \times Ac$$

$$\therefore Ac = \frac{Area\ AB \times AO - Area\ CD \times Ao}{Area\ AB - Area\ CD}.$$

The same formula will apply to lines and volumes by simply substituting *line* or *volume* for *area.*

225. Irregular Figures.—Any figure may be divided into rectangles and triangles, and, the centre of gravity of each being found, the centre of gravity of the whole may be determined by treating it as if it were an aggregation of particles, as in Articles 215 and 216.

226. The Centre of Gravity of a Zone *is at the middle point of the line joining the centres of the upper and lower bases of the zone.*

It is proved in Geometry that, on the same or equal spheres, zones are to each other as their altitudes; hence, if the zone be divided into an indefinite number of parallel zones, and all be reduced, or contracted, to the axis of the zone, it will form a line of uniform weight; and hence, the centre of gravity will be at the middle point of the line.

EXAMPLES.

1. If a line be drawn parallel to the base of a triangle, dividing it into equal areas, will it pass through the centre of gravity of the triangle?

2. If a line bisects the vertical angle of a triangle, in what cases will it pass through the centre of gravity, and in what cases will it not?

3. If the bases of two triangles are in the same line, and their vertices are in a line parallel to the bases, show that the line joining their centres of gravity will also be parallel to the bases.

4. In Fig. 90, if the circles are tangent to each other internally at A, find the distance from A to the centre of gravity c, after the smaller circle has been removed. Let $R = AO$; $r = Co$.

$$Ans.\ Ac = \frac{R^3 - r^3}{R^2 - r^2}.$$

5. Find the centre of gravity of the remainder of a square after one-quarter of it has been removed from one corner.

$$Ans.\ AB = \frac{5}{6} AC.$$

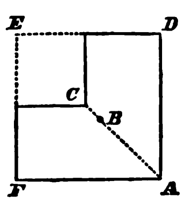


FIG. 91.

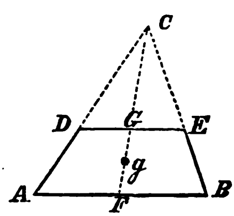


FIG. 92.

6. Find the centre of gravity of a trapezoid.

It will be on the line joining the centres of the two bases.

$$Ans.\ Fg = \tfrac{1}{3} FG \frac{AB + 2DE}{AB + DE}.$$

7. Find the centre of gravity of the surface of a right cone having a circular base, including the base. Let r be the radius of the base and h the altitude; find the distance from the apex.

$$Ans. \frac{\frac{2}{3}\sqrt{r^2+h^2}+r}{\sqrt{r^2+h^2}+r}h.$$

Centre of Gravity of Volumes.

227. Triangular Pyramid.—*The centre of gravity of any triangular pyramid is on the line joining any apex with the centre of gravity of the opposite face, and at a point three-fourths the length of the line from the apex.*

Let $A-BCD$ be a triangular pyramid. Suppose that it is divided into infinitely thin slices, bcd, parallel to the base, BCD. The centre of gravity of the pyramid will be on the line passing through the centres of all the slices. Let F be the centre of gravity of the base, then will the centre of gravity of the pyramid be on the line AF. Similarly, it will be on the line BG drawn from the apex B to the centre of gravity of the opposite face, and hence, at their intersection H. Join F and G, and the similar triangles FGE and AEB give

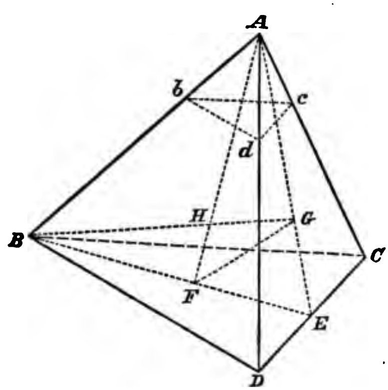


FIG. 93.

$$\frac{EG}{AE}=\frac{FG}{AB}.$$

But, $$EG = \frac{1}{3}AE;$$
$$\therefore FG = \frac{1}{3}AB.$$

The similar triangles, FGH and AHB, give

$$\frac{FG}{AB} = \frac{FH}{AH},$$

in which, substitute the value of FG, and it gives

$$3FH = AH.$$

Add $$FH = FH,$$

and we have $$4FH = AF;$$
$$\therefore FH = \frac{1}{4}AF.$$

228. The Centre of Gravity of any Pyramid or Cone *is on the line joining the apex with the centre of gravity of the base and at one-fourth the distance from the base.*

That it will be on this line is evident from the preceding Article. The pyramid may be divided into triangular pyramids, and the centre of each will be in a plane parallel to the base and at one-quarter the altitude from the base; hence, it will be at the point where this line intersects the plane. The position for the cone is found in the same way, for, the cone may be considered as composed of an indefinite number of pyramids.

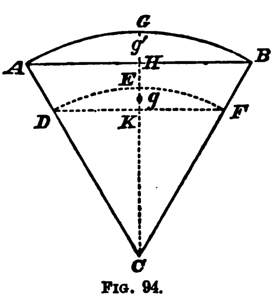


FIG. 94.

229. Problem. — *Find the centre of gravity of a spherical sector generated by the revolution of the circular sector AGC about the axis GC.*

It will be on the axis GC. If we consider that the spherical sector is composed of an indefinite number of cones, having their bases in the surface of the sphere, and their apices a

the centre C of the sphere, the locus of the centre of gravity of all the cones will be in a spherical surface DEF, having C for a centre and radius $CD = \frac{3}{4}$ of CA. The centre of gravity g of this surface will be the centre of gravity of the spherical sector; but, according to Article 226, Eg is one-half of the altitude EK of the spherical surface DEF. But, from the figure, we have

$$EK = \tfrac{3}{4}GH;$$
$$\therefore Eg = \tfrac{3}{8}GH;$$

and

$$\begin{aligned} Gg &= GE + Eg \\ &= \tfrac{1}{4}GC + \tfrac{3}{8}GH \\ &= \tfrac{1}{4}(GC + \tfrac{3}{2}GH). \end{aligned}$$

But,

$$\begin{aligned} Cg &= CG - Gg \\ &= CG - \tfrac{1}{4}CG - \tfrac{3}{8}GH \\ &= \tfrac{3}{8}(2CG - GH). \end{aligned}$$

230. Problem.—*To find the centre of gravity of a segment of a sphere.*

Let AGB, Fig. 94, be the segment of a sphere, and a point g' on the line GH be its centre of gravity. Taking the origin of moments at the centre C, we have

$$\begin{aligned} Vol.\ of\ Seg. \times Cg' = Vol.\ of\ sector\ AGBC \times Cg \\ - Vol.\ of\ cone\ ABC \times \tfrac{3}{4}CH; \end{aligned}$$

or,

$$\begin{aligned} \pi(GH)^2(CG - \tfrac{1}{3}GH).Cg' = 2\pi(CG)^2.\frac{GH}{CG}.\tfrac{1}{3}GC.Cg \\ -\pi(AH)^2.\tfrac{1}{3}CH.\tfrac{3}{4}CH; \end{aligned}$$

$$\therefore Cg' = \frac{8(CG)^2.GH.Cg - 3(AH)^2.CH^2}{12(GH)^2.(CG - \frac{1}{3}GH)}.$$

EXAMPLES.

1. Find the centre of gravity of a hemisphere.

   It will be on the radius perpendicular to the base of the hemisphere, and, according to Article 229, at a distance $\frac{3}{8}$ *the radius* from the centre.

2. Find the centre of gravity of the remainder of a sphere whose radius is R, after another sphere, whose radius is r, is taken from it, the two spheres having a common tangent plane.

3. Find the distance from the centre of a sphere to the centre of gravity of a segment of the sphere of one base, the chord of the segment being $\frac{1}{2}$ the radius.

4. A cone is suspended at a point in the circumference of the base; required the inclination θ of the axis to the horizontal. Let the radius of the base be 2 inches and the altitude 8 inches.

5. In the preceding example, what will be the relation between the radius of the base and altitude of the pyramid, if the axis is inclined 30°?

Centrobaric Method.

231. The two following theorems are by some accredited to Guldinus and by others to Pappus, one or the other of whom is supposed to have discovered them.

232. Theorem I.—*The surface, generated by the revolution of a line about an axis fixed in the plane of the line, is equivalent to the product of the length of the line into the circumference passed over by the centre of gravity of the line.*

Let AN be a plane curve, and YY the fixed axis in the plane of the curve. Draw any number of equal

chords, AB, BC, etc., and from their middle points, a_1, a_2, etc., draw the perpendiculars a_1b_1, a_2b_2, etc., to the axis YY. The revolution of the curve about the axis will generate a double curved surface, and the polygon, several frustra of cones inscribed within the former surface.

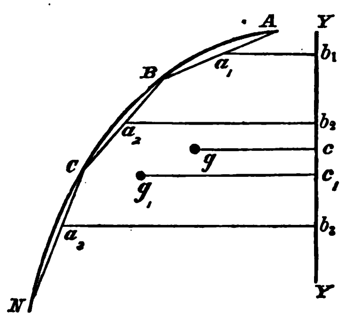


FIG. 95.

The surface generated by AB will be (see Geometry)

$$2\pi a_1b_1 \times AB.$$

Similarly, the entire surface generated by the polygon will be

$$2\pi(a_1b_1 \times AB + a_2b_2 \times BC + \text{etc.}).$$

Let g be the centre of gravity of all the lines AB, BC, etc., and gc, a perpendicular to YY, then will the moments of the lines in reference to the axis YY be

$$gc(AB + BC + \text{etc.}) = AB.a_1b_1 + BC.a_2b_2 + \text{etc.}$$

Multiplying both members of this equation by 2π, gives

$$2\pi gc(AB + BC + \text{etc.}) = 2\pi(AB.a_1b_1 + BC.a_2b_2 + \text{etc.}),$$

the second member of which is the surface generated; hence,

$$2\pi gc \times \textit{perimeter of the polygon} = \textit{surface generated},$$

in which $2\pi gc$ is the circumference described by the centre of gravity of the perimeter.

Inscribe in the curve another polygon of double the number of sides, and so on indefinitely; the limit of the polygons is the arc, and the limit of the surface is the double curved surface; but, the preceding equation is true for any number of sides, and hence, will be true of

the limit. Let g_1 be the centre of gravity of the arc, and g_1c_1 the perpendicular upon the axis; then the equation becomes

$$2\pi g_1c_1 \times \textit{length of arc} = \textit{surface generated by the arc.}$$

The theorem is evidently true for a single line, or for several lines of unequal length. Q. E. D.

233. Theorem II.—*The volume, generated by the revolution of a plane area about a fixed axis in its plane, is equivalent to a prism whose base is the area revolved, and altitude, the length of the circumference passed over by the centre of gravity of the area.* The plane area must lie wholly on one side of the axis.

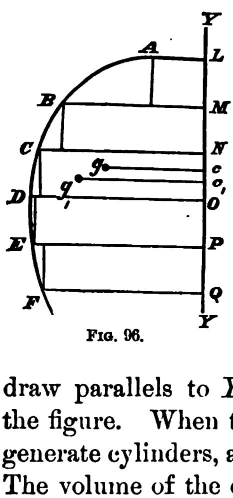


FIG. 96.

Let $LAB\ldots FQ$ be a plane area, YY the fixed axis in the plane of the area. Divide LQ into equal parts, LM, MN, etc., and draw the ordinates LA, MB, etc., perpendicular to YY, and from A, B, etc., draw parallels to YY, forming rectangles, as shown in the figure. When the figure revolves the rectangles will generate cylinders, and the curve, a double curved surface. The volume of the cylinders will be

$$\pi AL^2 . LM + \pi BM^2 . MN + \text{etc.}$$

Let g be the centre of gravity of all the rectangles, and the ordinate to the centre of gravity of each being equal to one-half the length of the side of the rectangle, we have for the moments of the rectangles in reference to the axis YY,

$$gc \times \textit{area of all the rectangles} = AL \,.\, LM \,.\, \tfrac{1}{2}AL + BM \,.\, MN \,.\, \tfrac{1}{2}BM + \text{etc.}$$

$\therefore$ *$2\pi gc \times$ area of all the rectangles* $= \pi AL^2 . LM + \pi BM^2 . MN +$ etc.

$=$ *Vol. of all the cylinders.*

If now the divisions in LQ be increased indefinitely, the limit of the rectangles will be the area of the curve, and the limit of the cylinders will be the volume of revolution. Let g_1 be the centre of the plane area, then we have

Vol. of revolution $= 2\pi g_1 c_1$ *. area of the plane curve,*

$=$ *the area of the curve* $\times$ *by the distance described by the centre of gravity of the area.* Q. E. D.

Applications.

234. Problem.—*To find the centre of gravity of a circular arc.*

Let ABC be a circular arc whose centre is O; the centre of gravity will be at some point c on the radius OB drawn to the middle point, B, of the arc. Through O draw the axis YY parallel to the chord AC, and conceive the curve to be revolved about this axis, generating a zone. The area of the zone will be (see Geometry)

$$2\pi OB . AC.$$

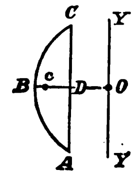


FIG. 97.

According to Theorem 1, we have

$$Arc\ ABC . 2\pi Oc = Area\ of\ the\ zone = 2\pi OB . AC;$$

$$\therefore Oc = \frac{AC . OB}{ABC}.$$

235. Problem.—*Find the distance from the centre of a circle to the centre of gravity of a sector of the circle.*

Let $AGBC$ be the sector generated by the rotation of the line AC about the centre C. Let θ be the angle ACG, and r the radius AC; then will the arc $AGB = 2r\theta$. According to Theorem 1 we find that the area of the sector will be $r^2\theta$.

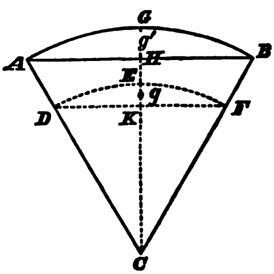


FIG. 98.

If the sector be divided into an indefinitely large number of sectors, each may be considered as a triangle whose centre of gravity is at two-thirds of its altitude from the centre C. With a radius DC, equal to $\frac{2}{3}r$, describe the arc DEF; this arc will be the locus of the centre of gravity of all the small sectors; and the centre of gravity of all of them, or the sector ACB will be at g, the centre of gravity of the arc DEF. According to Article 234 we have

$$Cg = \frac{2 \times \frac{2}{3}r \sin\theta \,.\, \frac{2}{3}r}{2 \times \frac{2}{3}r\theta} = \frac{2}{3}\frac{r\sin\theta}{\theta}$$

EXAMPLES.

1. If $ACBG$ is a semi-circle, prove that Cg is $\frac{4}{3}\frac{r}{\pi}$.
2. If g', Fig. 98, be the centre of gravity of the circular segment AGB, find the distance Cg'.
3. Find the volume of a sphere.
4. Find the volume generated by the revolution of the circular segment AGB about an axis through C and parallel to the chord AB.
5. Find the volume generated by the circular sector $CAGB$, about an axis through C and parallel to AB.

CHAPTER X.

SOLUTION OF PROBLEMS ACCORDING TO THE PRINCIPLES OF ENERGY.

Problems in which the Solutions depend upon Potential Energy.

236. Energy represented by the Three States of Equilibrium.—According to Article 210, when a body is in the condition of unstable equilibrium, the centre of gravity is in the highest position. In this condition its *potential energy* is a maximum, that is, it is in a condition to do the most work. When the centre of gravity is lowest, the body is in a condition of *stable equilibrium* (Article 209), and its *potential energy* is a *minimum*, that is, it is in a condition to do the least work. In neutral equilibrium, the *potential energy*, for successive positions of the body, *remains constant.*

237. *To find a curve such that a heavy bar AE, resting against it and against a vertical plane DE, will be in equilibrium in all positions, there being no friction on the surfaces.*

FIG. 99.

Let g be the centre of gravity of the bar. If the bar is prismatic and homogeneous, g will be at the middle of the length, but, in other cases, it may be at any other point along the bar.

This is a case of indifferent equilibrium, and hence, the centre of gravity is neither raised nor lowered by a change

of position; in other words, its locus will be in a horizontal line. Therefore, if the centre of gravity g be moved along the line CB, and the end E be kept constantly against the vertical DE, the end A will trace the curve. But this is equivalent to the well-known method of constructing an ellipse by means of a trammel. Hence, the curve CAD is an ellipse.

[Let

$$AE = l \text{ and } Ag = \tfrac{1}{2}l;$$

$$x = BF,\ y = FA;$$

then,

$$Fg = \tfrac{1}{2}x,$$

and

$$FA^2 + Fg^2 = Ag^2;$$

or,

$$y^2 + \tfrac{1}{4}x^2 = \tfrac{1}{4}l^2;$$

which is the equation of the ellipse. A more general solution will be found by letting $Ag = n \,.\, Eg$.]

238. *Required the form of a curve such that a heavy bar resting against it and against a smooth pin above the curve, will be in equilibrium in all positions.*

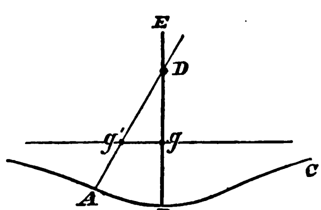


FIG. 100.

Let AD be the bar, D the position of the pin, and ABC the required curve.

This is also a case in which the *potential energy is constant;* hence, the centre of gravity will be found in a horizontal line gg', passing through the centre of gravity, g, of the bar in the vertical position. The curve may therefore be constructed by drawing any number of radial lines DB, DA, etc., through D, intersecting them by the horizontal line gg', and laying off on the radial lines below

the horizontal the constant distance $gB = g'A$, etc. The curve is called the *conchoid of Nicomedes.*

[Let

$$Ag' = Bg = a;\ gD = c;\ ADB = \theta;\ AD = \rho;$$

then,

$$AD = Ag' + g'D$$

$$= Ag' + Dg \sec\theta;$$

or,

$$\rho = a + c \sec\theta;$$

which is the polar equation of the curve.]

239. *A cord of given length is suspended at two points in the same horizontal; required the form of the curve when the centre of gravity is the lowest.*

The cord, being perfectly flexible, will naturally assume the position of stable equilibrium, and its potential energy will be a minimum; that is, its centre of gravity will be the lowest possible. The curve assumed by such a cord is called a *Catenary.* If the cord be of variable density, it will still assume the position in which the centre of gravity is lowest.

FIG. 101.

[The equation of the catenary is found by higher mathematics. If w = the weight per unit of length of the cord; t_0 = the tension at the lowest point of the cord; ε = the base of the Naperian system of logarithms; x horizontal and y vertical; the origin of coördinates being taken at the lowest point, then

$$y = \tfrac{1}{4}\frac{t_0}{w}\left[\epsilon^{\frac{wx}{2t_0}} - \epsilon^{-\frac{wx}{2t_0}}\right]^2.$$

240. *A curve of given length is revolved about the line passing through its extremities; required the form of the curve such that the surface generated shall be a maximum.*

The area will equal the length of the curve multiplied by the distance passed over by the centre of gravity of the curve. The curve being of constant length, the area will therefore be greatest when the distance of the centre of gravity of the curve from the axis of revolution is greatest; hence the curve must be a catenary.

241. *Two cylinders of unequal radii, but of the same material and length, are placed in a larger hollow cylinder; find their position when in equilibrium.*

Their position will be that in which the potential energy is least; hence, their centre of gravity will be lowest.

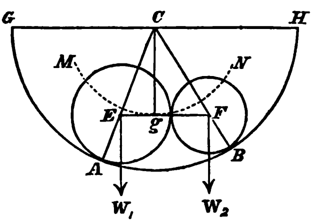


FIG. 102.

Let g be the centre of gravity of the cylinders when in contact. If they be rolled in the hollow cylinder, remaining in contact with each other, the centre of gravity will describe a circle MgN, about the centre C of the hollow cylinder, and the line EF joining the centres of the cylinders will be constantly tangent to this arc; hence, when they are in the position of equilibrium, the line EF will be horizontal and the point g vertically under the centre C.

To find the angle ECg, let $R = AC$, $r_2 = FB$, and $r_1 = AE$.

Then, $EF = r_1 + r_2 =$ the sum of the radii of the two cylinders; and the equation of moments gives

$$W_1 . Eg = W_2 . gF;$$

and from the figure we have

$$Eg + gF = r_1 + r_2$$

$$\therefore Eg = \frac{W_2}{W_1 + W_2}(r_1 + r_2);$$

$$CE = R - r_1$$

$$\therefore \sin ECg = \frac{W_2(r_1 + r_2)}{(W_1 + W_2)(R - r_1)}.$$

?equired the effort necessary to maintain a body 'oth, inclined plane, the effort being exerted par- he plane.

? effort be exerted by a body acting vertically, as 03; then will the centre of gravity of the two ; in the same horizontal line for all positions of on the plane.

C be the inclined plane, on which the weight P ι position by the weight W, all without friction. 1 d be the centres of the ?spectively in one posi- n will their centre of ?e at some point g, on joining their centres. dies are moved into an- ition, having their cen- ?ctively at b and e, their f gravity will be in a .l line through g.

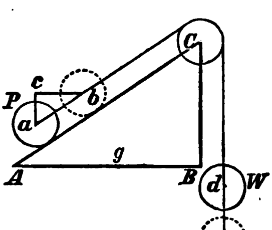


FIG. 103.

he energy is constant, the potential energy gained g the weight P will equal that lost by lowering other words, the product of P into the vertical through which it has been raised, equals W into ?al distance through which it has been lowered.

Drawing the horizontal line bc, and the vertical line ae, we have

$$P.ac = W.de;$$

but,

$$\begin{aligned} ac &= ab \sin cba \\ &= ab \sin A \\ &= de \sin A \\ &= de \frac{BC}{AC}; \end{aligned}$$

which, substituted in the preceding equation, gives

$$P.BC = W.AC;$$

or,

$$P : W :: AC : BC;$$

that is, *the effort is to the resistance as the height of the plane is to its length.*

243. *Determine the conditions of equilibrium of a single pulley.*

In one position let the weight W be at d, and P, at a; and in another position, P at b, and W at c. The centre of gravity will be at the same point g, in both positions; hence, we have

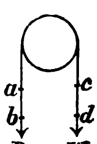


FIG. 104.

$$P.ab = W.dc;$$

but,

$$ab = dc;$$

$$\therefore P = W;$$

that is, *the effort equals the resistance when there is no friction.*

244. *Determine the conditions of equilibrium of the straight lever.*

Let AB be a bar in a horizontal position, having weights P and W suspended at its extremities; it is required to find the point C, upon which they will balance. Their

centre of gravity, g, will be in the line joining the centres of the bodies, and vertically under the required point C. Let the arm be turned into the position ab, the perpendiculars da and eb from a and b be dropped upon AB; then will the weight W have been raised a height equal to be, and P will have fallen a distance ad. Since there is equilibrium, we have

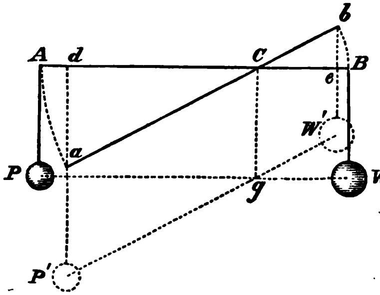


FIG. 105.

$$P.da = W.eb.$$

The similar triangles adC and beC, give

$$\frac{da}{eb} = \frac{aC}{bC} = \frac{AC}{BC};$$

and, by combining these equations, we find

$$P.AC = W.BC;$$

or,

$$P : W :: BC : AC;$$

that is, *the effort is to the resistance inversely as the arm of the effort is to the arm of the resistance.*

In this problem the centre of gravity of the bodies remains at g for all inclinations of the arm AB; for the similar triangles adC and Ceb, give

$$dC : Ce :: aC : Cb;$$

but dC and Ce are the arms of the forces in the new position; hence, they are proportional to the original arms. When the support is not in the line of the points of attachment of the weights, the centre of gravity will change its

position as the arm is rotated, as will be seen in the following problem:

245. *To find the conditions of equilibrium of the bent lever.*

Let AG and BG be the arms of the lever, the support, or fulcrum, being at G, and the weights suspended as shown in the figure.

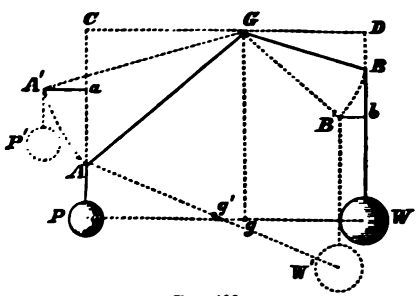


FIG. 106.

The centre of gravity of the weights, when in equilibrium, will be in the line joining the centres of P and W, and at a point g, vertically under the point of support G. Let the lever be turned through a small angle, then will the end A describe the arc AA', and B, the arc BB', and the weights will be found in the positions P' and W'. The centre of gravity will be *raised* to a point g'; hence, when the bodies are left to themselves they will return to their former position.

We may, however, determine a practical formula for this case by assuming that they remain in equilibrium when the lever is turned through an exceedingly small angle. For this case, we consider the arcs AA' and BB' as straight lines. Draw the horizontal lines $A'a$ and $B'b$, then will the potential energy of P be increased by an amount equal to

$$P.Aa;$$

and that of W will be diminished by an amount equal to

$$W.Bb;$$

hence, according to the hypotheses, we have

$$P.Aa = W.Bb.$$

Through G draw the horizontal CD, meeting the verticals through A and B at the points C and D. From the similar right angled triangles $AA'a$ and ACG, we have

$$\frac{Aa}{AA'} = \frac{GC}{GA}.$$

Similarly, the triangles $BB'b$ and BGD give

$$\frac{Bb}{BB'} = \frac{GD}{GB}.$$

But the arcs BB' and AA' are proportional to the radii GB and GA; hence,

$$\frac{AA'}{GA} = \frac{BB'}{GB};$$

and, by combining these three equations so as to eliminate Aa and Bb, we find

$$P.GC = W.GD;$$

or,

$$P : W :: GD : GC;$$

that is, *the weights are inversely as their arms;* a result which agrees with the preceding problem.

Problems involving Kinetic Energy.

246. *A body falls freely through a height h; what will be the kinetic energy stored in it?*

A body whose weight is W, at a height h above a given point, has a potential energy of

$$Wh;$$

and when it has fallen through this height its energy will

be changed to kinetic energy. Substituting for h its value in terms of v (see Eq. (3), Art. 72), gives

$$K = W\frac{v^2}{2g} = \tfrac{1}{2}Mv^2,$$

as given in Article 112.

If the body had fallen through a portion of the height h_1, leaving a height h_2, through which it may afterwards fall, we have the kinetic energy

$$K = Wh_1 = \tfrac{1}{2}Mv_1^2,$$

and the potential energy

$$\Pi = Wh_2;$$

hence, the total energy will be

$$\begin{aligned} K + \Pi &= W(h_1 + h_2) \\ &= Wh, \end{aligned}$$

which is constant for that height.

247. *Two bodies of unequal weights are placed on two unequally inclined planes, and connected by a fine inextensible cord, which passes over a pulley so placed above the angle of the planes that the cord will be parallel to the planes; required the equations for their motion, there being no frictional resistances, nor resistance of the air.*

Let AC and BC be the inclined planes, W and P the positions of the bodies when motion begins. After a time t let them be in the positions represented by P' and W'; the distances over which they have moved being

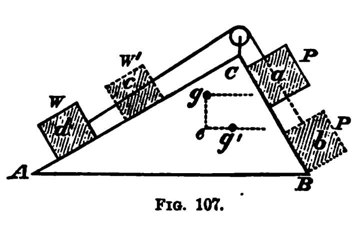


FIG. 107.

$$dc = ab.$$

In the initial position, the centre of gravity of the bodies

will be at some point g, in the line joining their respective centres of gravity; and after a time t it will be at some lower position g'. Through g and g' draw horizontal lines; the vertical distance ge between them, will be the height through which the common centre of gravity of both bodies will have fallen.

Let

$$ge = h;$$

then, the potential energy lost will be

$$(P + W)h,$$

which is a gain of kinetic energy equal to

$$(P + W)\frac{v^2}{2g}.$$

Let

$$s = ab = dc;$$

then will the vertical height through which P has fallen be

$$s \sin B;$$

and the height through which W has been raised will be

$$s \sin A;$$

and the total potential energy lost will be

$$(P \sin B - W \sin A)s;$$

hence, we have

$$(P + W)\frac{v^2}{2g} = (P \sin B - W \sin A)s;$$

from which we find

$$v^2 = \frac{P \sin B - W \sin A}{P + W} 2gs. \quad . \quad . \quad (1)$$

In this problem the acceleration will be constant; hence, according to Article 24, we have

$$s = \tfrac{1}{2}vt;$$

which, substituted in the preceding and reduced, gives

$$v = \frac{P \sin B - W \sin A}{P + W} gt. \quad . \quad . \quad . \quad (2)$$

Eliminating v from the two preceding equations, gives

$$t = \sqrt{\left[\frac{P + W}{P \sin B - W \sin A} \cdot \frac{2s}{g}\right]}. \quad . \quad (3)$$

248. *If one of the bodies as P, in the preceding problem, moves vertically, while the other moves on the plane, required the formulas for their motion.*

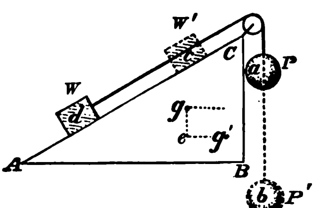


FIG. 108.

This problem may be solved in the same way as the preceding one; but the results may be obtained directly from the preceding formulas by making $B = 90°$. The results are

$$v = \sqrt{\left[\frac{P - W \sin A}{P + W} 2gs\right]}.$$

$$= \frac{P - W \sin A}{P + W} gt;$$

and,

$$t = \sqrt{\left[\frac{P + W}{P - W \sin A} \cdot \frac{2s}{g}\right]}.$$

249. *If, in the preceding problems, both bodies move vertically, required the formulas for the motion.*

Making $A = 90°$, in the preceding formulas, we find the formulas of Problem 3, Article 90.

250. *A vessel is filled with a liquid; required the velocity with which it will discharge itself through an orifice near the base.*

Let ABE be the vessel, F the position of the orifice. Suppose that a small portion, equal to the horizontal slice ACB, has been discharged. The centre of gravity of the mass will have been lowered from some point g to another point g', and the potential energy lost is equal to the weight of the liquid above the orifice, multiplied by the distance gg'. But a more simple way of considering the problem, is to assume that the centre of gravity of the part below the slice ACB remains at g', and hence, that the change in the position of the centre of gravity has been produced by the transference of the slice ACB to the level of the orifice F. Assuming that there are no frictional resistances, nor resistance from the air, then the whole energy will be expended in producing the motion of the liquid.

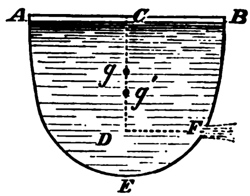


FIG. 109.

Let

$S =$ the horizontal section of the vessel at AB,

$k =$ the section of the orifice at F,

$x =$ the thickness of the thin slice AB,

$h = DC =$ the vertical height of AB above the orifice,

$t =$ the time of the discharge of a quantity of the liquid equal to that in the slice AB,

$w =$ the weight of a unit of volume—say one cubic inch of the liquid, and

v = the velocity of the discharge.

Then,

Sx = the volume of the slice,

wSx = the weight of the slice, and

$wSx\,h$ = the work accumulated in the slice when i
fallen through the height h.

The quantity which will flow through the orifice
time t will be

$$kvt,$$

the weight of which will be

$$wkvt,$$

the mass of which is

$$\frac{wkvt}{g},$$

and of which the kinetic energy, due to the flow, wi

$$\tfrac{1}{2}\frac{(wkvt)}{g}v^2;$$

hence,

$$(wSx)h = \tfrac{1}{2}\frac{(wkvt)}{g}v^2;$$

But the weight $wSx = wkvt$, and by cancelling an
ducing we have

$$v^2 = 2gh;$$

which is the same as that of a particle falling f
through a height h.

This result is modified in practice on account of th
sistance of the air, friction, and viscosity of the liquid
there is a pressure upon the top of the vessel, we asce
what height of the liquid will produce the same pres
per square inch of the upper surface, and add the he
to the value of h, given in the problem. The he
which induces the flow is called *a head.*

EXAMPLES.

1. If, in Fig. 107, $P = 25$ lbs., $W = 30$ lbs., the angle $B = 60°$, and $A = 30°$, determine which will move down the plane, P or W, and how far they will move in 5 seconds.
2. Determine the relations between the weights P and W, and the angles A and B for equilibrium in Fig. 107.
3. If the angle $A = 30°$ and $B = 45°$, find the relation between P and W so that the acceleration of the bodies will be $\frac{1}{8}$ that of a body falling freely.
4. In Fig. 102, if the radius CA is 3 feet, AE 1 foot, and FB 6 inches, and the internal cylinders of the same material, what will be the angle ECg for equilibrium?
5. Solve the problem in Article 248, when there is a constant frictional resistance on the plane equal to $\mu W \cos A$.
6. In the problem of Article 241, find the angle ECg, where the cylinders E and F have the same diameter, but $W_1 = 2\,W_2$.
7. A vessel is filled with a liquid and kept constantly full; required the time necessary for the discharge of a quantity q from an orifice whose section is k, the distance of the orifice below the surface of the liquid being h feet.

CHAPTER XI.

CONSTRAINED EQUILIBRIUM.

251. A body is said to be *constrained* when it is prevented from moving in a particular direction on account of a point or line of the body being *fixed*, or on account of the interposition of a body which is considered as *immovable.* By the term *fixed* is not meant that the resistance offered by a point, line, or surface, cannot be overcome by any force, but that it will not be equaled by any force which may be involved in the problem. The term *immovable* is also considered in the same restricted sense. When only a point of the body is fixed, the body will be free to rotate about it in all directions; and when a line is fixed the body may rotate about it or slide along it.

252. Normal Resultant.—If a body be in equilibrium on a smooth surface, the resultant of all the forces which act upon it must be in the direction of the normal to the surface and act towards the surface. For, if it be not normal, it may be resolved into two components, one of which will be normal and the other tangential, the latter of which would produce motion.

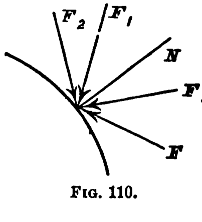


FIG. 110.

253. Equilibrium of a Body on a Smooth Inclined Plane.—Let AC be the inclined plane, o the centre of the body, F the resultant of all the forces which act upon the body, and W the weight of the body. Resolve the weight

W into two components, one, cb, parallel to the plane, and he other, ob, normal to it; and, similarly, resolve the orce F into the normal com- ›onent de, and the component ›e parallel to the plane. The ›xcess of ob over de need ›ot be considered, since the plane must resist it; but, that here shall not be movement long the plane, the compo- ient cb must equal oe.

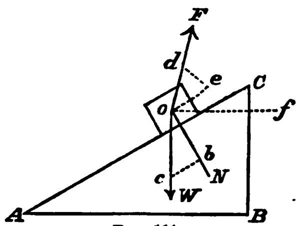


FIG. 111.

From the figure we have

$$cb = oc \sin cob$$
$$= W \sin A;$$

ɑd

$$oe = od \cos doe$$
$$= F \cos \phi,$$

here $\phi = doe$, the angle between the action-line of the ›rce and the plane, hence

$$F \cos \phi = W \sin A;$$

$$\therefore \cos \phi = \frac{W}{F} \sin A.$$

254. Equilibrium on a Rough Inclined Plane.— et the notation be as in the preceding article. The mount of friction will be, according to Articles 107 and 08, the normal pressure multiplied by the coefficient of riction, or,

$$\mu(ob - de)$$
$$= \mu(W \cos A - F \sin \phi).$$

If the body is in a state bordering on motion down the lane, the upward pull of F along the plane, and the riction, will equal the downward pull of W; or,

$$F\cos\phi+\mu(W\cos A-F\sin\phi)=W\sin A;$$

from which we find

$$F=\frac{\sin A-\mu\cos A}{\cos\phi-\mu\sin\phi}W.$$

If the value of F is such that the body is in a state bordering on motion *up* the plane, the component of F, parallel to the plane, will equal the component of W, also parallel to the plane, *plus* the friction; or

$$F\cos\phi=W\sin A+\mu(W\cos A-F\sin\phi)$$

$$\therefore F=\frac{\sin A+\mu\cos A}{\cos\phi+\mu\sin\phi}W.$$

255. *Determine the conditions of equilibrium of a homogeneous disc, having a hole cut in it near one edge, the disc being placed vertically on an inclined plane which is sufficiently rough to prevent sliding.*

This problem is essentially the same as if one side of the disc were loaded with a heavier substance, or if, from any other cause, the centre of gravity is not at the centre of the disc.

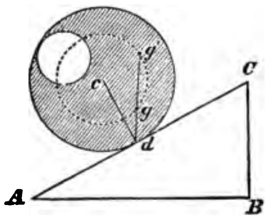


FIG. 112.

Let g be the centre of gravity of the body. The point of support for equilibrium must be at d vertically under g, which will also be a point of tangency of the circle and plane.

With c as a centre, and radius cg, describe a circle; then if the centre of gravity be at g' vertically over d, the body will *also be in equilibrium.* In the former position the equilibrium will be stable; in the latter, unstable. In the latter case, if the body be dis

turbed, it may appear to roll up the plane, but the centre of gravity will really be falling until it assumes a position of stable equilibrium. If the vertical through d falls outside the circle gg', it will not be in equilibrium in any position; if tangent to it, the body will be in equilibrium in only one position.

256. *To find the inclination of the plane so that the unbalanced disc of the preceding problem will just be on the point of rolling down the plane.*

According to the conditions given in the preceding article, the vertical through d must be tangent to the circle gg'. Through c draw a horizontal line, and a right-angled

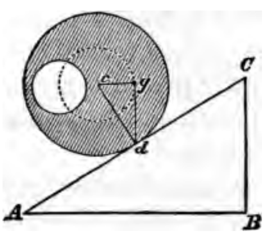


FIG. 113.

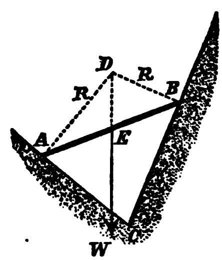


FIG. 114.

triangle cgd, will be formed, in which the angle cdg, Fig. 113, will equal CAB; hence,

$$\sin A = \sin d = \frac{cg}{cd}$$

$$= \frac{\textit{dist. of centre of gravity from centre of circle}}{\textit{radius of the circle}}.$$

257. *A small prismatic bar rests upon two smooth inclined planes; determine the position for equilibrium.*

Let AB be the beam, and E its centre of gravity. The

reactions at A and B must be perpendicular to the respective planes; and since these reactions, and the weight W, are the only forces which act upon the body, their lines of action must all meet in a common point. Hence, the perpendiculars AD and BD must meet the vertical through E at a common point. If a string ADB be attached to the extremities, A and B, of the bar, and hung on a smooth pin at D, vertically over the centre of gravity E, there will be no tendency to slide on the pin, and the bar will remain at rest in the inclined position. If the inclinations of the planes are given, a formula may be found for the inclination of the bar.

258. *A prismatic bar rests upon the edge of a smooth hemispherical bowl, one end being against the inner surface of the bowl; required the position for equilibrium.*

Let E be the centre of gravity of the bar, and C the centre of the bowl. At A the reaction will be normal to the surface of the bowl; and hence, its direction will coincide with the radius and pass through the centre C. At F the reaction will be perpendicular to the bar; hence, when the radius through A prolonged, meets the perpendicular FD, at a point D in a vertical through E, the bar will be in equilibrium.

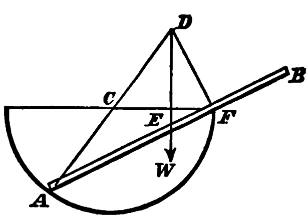


FIG. 115.

259. *A weight W is on the arc of a vertical circle whose centre is O, and is held in position by a weight P suspended from a cord, which passes over a pin at C, vertically over O; required the position for equilibrium, there being no frictional resistances.*

The tension upon the string will be equal to the weight P; hence, the body W will be held by a force equal to P acting in the direction cC, the force of gravity acting vertically downward, and the normal reaction of the curve. The normal to the circle coincides with the radius passing through the point, and hence, passes through the centre of the circle.

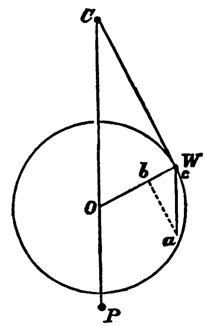


FIG. 116.

Draw a vertical ca to represent the weight W, and ab, parallel to cC, to represent P; then the position must be such that the extremity b, of the line ab, will fall on the normal cO. The triangles abc and OCc are similar, hence, we have

$$W : P :: ca : ab$$
$$:: OC : Cc;$$
$$\therefore Cc = \frac{P}{W} OC.$$

260. *A particle W, attached to one end of a string, is placed on the convex side of a smooth parabola, and held in position by a weight P attached to the other end of the string, the string passing over a smooth pin at the focus of the parabola; required the position for equilibrium.*

Let the axis of the parabola be vertical, C the focus, and c the position of the weight W. Construct the triangle of forces cba as before, ce being the normal to the parabola. Since ca is parallel to the axis Ce, the normal will bisect the angle Cca; hence,

$$Ccb = bca;$$

but, from the construction,

$$abc = Ccb;$$

$$\therefore abc = bca;$$

therefore, the triangle abc must be isosceles, and we have

$$ab = ac,$$

or,

$$W = P,$$

which establishes a relation between the given quantities. It does not determine any particular point, but it shows

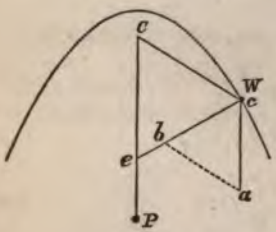


FIG. 117.

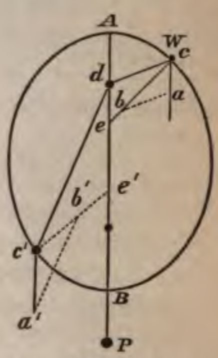


FIG. 118.

that, in order that the weights shall be in equilibrium at any point, they must equal each other, in which case they will be in equilibrium at all points on the curve.

261. *Instead of a parabola, let the curve be an ellipse, the major axis being vertical and the pin at the upper focus.*

Construct the triangle of forces cab as before, ce being the normal. Then,

$$W : P :: ca : ab$$

$$:: de : dc.$$

Let e be the eccentricity of the ellipse, then it is shown in Conic Sections, that

$$de : dc :: e : 1;$$

hence,

$$W : P :: e : 1;$$

or,

$$\frac{W}{P} = e;$$

that is, when the ratio of W to P equals the eccentricity of the ellipse, they will be in equilibrium for all positions of W on the curve; and if they have any other ratio they will not be in equilibrium at any point, except at the extremities of the axis, A and B.

[In "Conic Sections" the proportion involving e, given in this article, may not be given in this form. If a be the semi-major axis, and x the abscissa of the point c in reference to the centre, then the following equations are usually given, viz.:

the radius vector, $dc, = a \pm ex$;
the distance from the focus to the foot of the normal,
$de, = e(a \pm ex)$;

the *plus* sign being used for the distance $e'd$, and the *minus* sign for ed. These values readily give

$$\frac{de}{dc} = \frac{e(a \pm ex)}{a \pm ex} = e;$$

hence, the proportion given in the text.]

262. *Let the curve be an hyperbola, the pin being in the upper focus.*

Proceeding as before, we find

$$\frac{W}{P} = e;$$

but in this case e exceeds unity; hence W must exceed P, while in the former case W must be less than P.

263. *Suppose, in the preceding case, that the pin on which the string passes is at the centre of the hyperbola.*

Let a be the semi-major axis, b the semi-minor axis, t

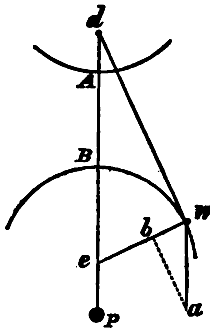


FIG. 119.

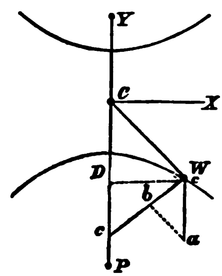


FIG. 120.

axis of x horizontal, and y vertical. The axis of the hyp bola being vertical, the equation will be

$$a^2x^2 - b^2y^2 = - a^2b^2. \quad . \quad . \quad . \quad (1)$$

Construct the triangle of forces cab, in which ab will parallel to cC; draw cD horizontal, and we have

$$W : P :: ca : ab$$
$$:: Ce : Cc;$$

which, according to the principles of Conics, becomes

$$W : P :: e^2.CD : Cc;$$

or, observing that $CD = y$, and $Dc = x$, we have

$$W : P :: e^2y : \sqrt{y^2 + x^2};$$

hence

$$W^2 : P^2 :: e^4y^2 : y^2 + x^2;$$

which, reduced to an equation, gives

$$W^2y^2 + W^2x^2 = P^2e^4y^2. \quad . \quad . \quad (2)$$

According to Conics we also have

$$e^2 = \frac{a^2 + b^2}{a^2}. \quad . \quad . \quad . \quad (3)$$

Eliminating a and x from these three equations, we find

$$y = \frac{bW}{e\sqrt{W^2 - e^2P^2}} = CD$$

$$= \frac{b}{e\sqrt{1 - e^2\frac{P^2}{W^2}}}.$$

EXAMPLES.

In Fig. 111, if $W = 50$ lbs., $A = 30°$, $Fof = 60°$, what must F be for equilibrium?

In Fig. 121, if $CAB = 45°$ $W = 60$ lbs., what must F be, acting horizontally, for equilibrium? (The value of F may be found from the equation of Article 253, observing that ϕ will be $-45°$.)

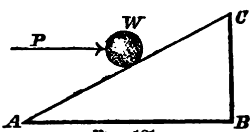


FIG. 121.

Find the value of F, Fig. 111, for equilibrium when it acts parallel to the plane. (Deduce it from the equation of Article 253.)

Similarly, find the value of F, for equilibrium, when it acts horizontally.

In Fig. 111, if $A = 30°$, $W = 50$ lbs., coefficient of friction $\mu = 0.2$, the angle $\phi = -5°$; what will be the value of F so as to just prevent motion down the plane; also so that it shall be just on the point of moving it up the plane?

6. In the same figure, if $A = 45°$, $W = 100$ lbs., $\mu = 0.15$, what will be the value of F, acting parallel to the plane, so as just to prevent motion down the plane.
7. In Fig. 113, if the diameter of the small circle equals the radius of the larger one, and the remainder of the larger one be homogeneous, what will be the greatest inclination of the plane AC that the body may remain at rest, there being no sliding?
8. The weight W being 20 lbs., P 15 lbs., in Fig. 116, the radius of the circle 2 feet, the distance OC, from the centre O to the pin, being 4 feet; required the angle OCc for equilibrium?
9. In Fig. 116, if $OC = Cc =$ the radius of the circle, what will be the relation of P and W, and the angle COc, for equilibrium.
10. In Fig. 120, if $W = 100$ lbs., $P = 25$ lbs., $b = 3$ feet, and $e = 2$, what will be the distance CD for equilibrium.

EXERCISES.

1. If a force, equal to the reaction of the plane, be substituted for the plane, will the body remain in equilibrium?
2. If a plane is perfectly smooth, can a body remain at rest on it if the plane has any inclination? If the body were pressed *normally* against the plane, would it remain at rest?
3. In Fig. 112, what will be the position of the disc so that the potential energy shall be greatest, and what, so that it shall be least? In Fig. 113, is the potential energy a maximum, a minimum, or indifferent?
4. Is the potential energy of the bar in Fig. 114 a maximum or a minimum? Similarly, in regard to the bar in Fig. 115?
5. When the relations of P to W are such as to secure equilibrium in Figs. 117, 118, and 119, will the potential energy be a maximum, a minimum, or indifferent?

6. In Article 263, when the bodies are in equilibrium, will the potential energy be a maximum or a minimum?
7. In Article 262, can the bodies be in equilibrium on the curve, if P exceeds W?
8. In Fig. 118, if W exceeds P, can there be equilibrium at any point of the curve?
9. In Fig. 115, what is the longest bar that can rest on the bowl and have one end, A, in the bowl.
10. If the planes are equally inclined in Fig. 114, what will be the position of the bar when in equilibrium?
11. If the plane AC, Fig. 114, were horizontal and perfectly smooth, what must be the position of the bar for equilibrium?

CHAPTER XII.

ANALYTICAL METHODS.

264. Analytical Mechanics is generally understood to refer to that system of **analysis in which the equations of** equilibrium, or of motion, **are established in reference to** a system of coördinates, and the **results obtained by opera**tions upon those equations. **The coördinates most com**monly used are Rectangular or Polar.

265. Before establishing the general equations, we observe that when the forces are in a plane they may be

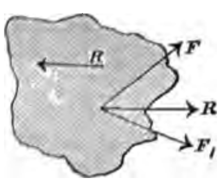


FIG. 122.

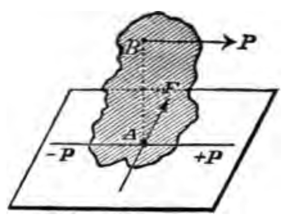


FIG. 123.

reduced to a single resultant, or to a single couple, Article 194. If they are not in one plane, let P, Fig. 123, be a force applied at B, and F another force applied at A. At any point A, of the force F, introduce two equal and opposite forces, each equal and parallel to P, and as they mutually destroy each other they will not change the mechanical condition of the problem. Now, combining P at B with $-P$ at A, we have a couple, whose moment is

$$P . AB;$$

and the other forces, F and $+P$, at A, will have a single

tant. A similar operation may be performed when
are several forces; hence, we *infer* that, when the
es are not in a plane, they may be reduced to a single
and a single couple. This result, however, may be
ved from the following equations.

6. **General Case.**—Forces may be applied at any or
he points of a body, and act in all conceivable direc-
. The angles which the
n-lines of the forces make
the axes may be determined
n Article 157. The origin
oördinates may be taken at
point, and the rectangular
have any position in refer-
to the body or the forces.
axis of x will be considered
ositive to the right, y posi-
upward, and z positive in front of the plane yx.

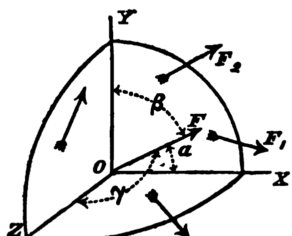


FIG. 124.

7. **Forces Resolved.**—Let the coördinate axes be
angular, and the forces resolved parallel to them.

F_1, F_2, F_3, etc., be the forces,

a_1, a_2, a_3, etc., the angles which the lines of the forces make respectively with the axis OX,

β_1, β_2, β_3, etc., the corresponding angles with OY,

γ_1, γ_2, γ_3, etc., the corresponding angles with OZ,

X, Y, Z, the algebraic sum of the components of the forces parallel respectively to the axes of x, y, and z,

R, the resultant of all the forces, and

a, b, and c, the angles which it makes with the axes x, y, z, respectively.

The components of the forces parallel to the axis of x will be, according to Article 156,

$$F_1 \cos a_1, \quad F_2 \cos a_2, \quad F_3 \cos a_3, \text{ etc., and}$$
$$R \cos a.$$

These components constitute a system of parallel forces, which are not confined to one plane, but are simply parallel to the axis of x. The resultant of any two of the forces, since *they* are in one plane, will be their algebraic sum, Articles 190 and 198; or,

$$F_1 \cos a_1 + F_2 \cos a_2.$$

This resultant and a third force will be in one plane, and, hence, the resultant of this resultant and a third force will be their algebraic sum; or,

$$F_1 \cos a_1 + F_2 \cos a_2 + F_3 \cos a_3.$$

Continuing this process, we finally have for the resultant of all the components parallel to the axis of x,

$$X = F_1 \cos a_1 + F_2 \cos a_2 + F_3 \cos a_3 + \text{etc.} = \Sigma F \cos a;$$

which value must equal the x-component of the resultant of all the forces; hence

$$X = R \cos a = \Sigma F \cos a.$$

Proceeding in a similar way with the y and z-components, we have

$$Y = R \cos b = \Sigma F \cos \beta,$$
$$Z = R \cos c = \Sigma F \cos \gamma.$$

Squaring and adding, we have

$$R^2 = X^2 + Y^2 + Z^2$$
$$= (\Sigma F \cos a)^2 + (\Sigma F \cos \beta)^2 + (\Sigma F \cos \gamma)^2.$$

The *direction* of the axis may be determined from the equations

$$\cos a = \frac{X}{R}, \quad \cos b = \frac{Y}{R}, \quad \cos c = \frac{Z}{R}.$$

If there is equilibrium we have

$$R = 0;$$
$$\therefore X = 0; \quad Y = 0; \quad Z = 0.$$

These equations are the same as those for concurrent forces, Article 161; hence, the tendency of a system of forces, having given intensities and directions of action, to produce translation will be the same whether they concur or not, and will be independent of the points of application of the forces.

268. Moments of the Resolved Forces.

Let

x_1, y_1, z_1 be the coördinates of the point of application of F_1, and a corresponding notation for the forces F_2, F_3, etc.

The component of F_1, parallel to the axis of x, will tend to turn the point of application either to the right or left about the axis of z (unless it be in the plane xz), and the arm of the component will be y_1; hence the moment will be, Article 164,

$$F_1 \cos \alpha_1 \,.\, y_1;$$

and the component of F_1, parallel to y, will also tend to turn the point of application about the same axis, z, and the moment will be

$$F_1 \cos \beta_1 \,.\, x_1.$$

In Fig. 125, the component $F_1 \cos \alpha_1$ tends to turn the

point of application right-handed, and $F_1 \cos \beta_1$, left-handed; hence, the resultant moment in reference to the axis of z will be

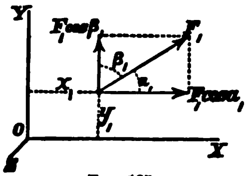


FIG. 125.

$$F_1(y_1 \cos a_1 - x_1 \cos \beta_1).$$

The quantity in the parentheses is the arm of the force F_1, as shown in Article 176. This expression is equivalent to

$$F_1 \cos a_1 . y_1 - F_1 \cos \beta_1 . x_1 ;$$

and the other forces give

$$F_2 \cos a_2 . y_2 - F_2 \cos \beta_2 . x_2$$
$$\text{etc.,} \qquad \text{etc.};$$

and taking the sum of these expressions, we have

$$\Sigma F \cos a.y - \Sigma F \cos \beta.x ;$$

or,

$$Xy - Yx ;$$

and, similarly, in reference to the axis of y, we have

$$Zx - Xz ;$$

and, in reference to the axis of x, we have

$$Yz - Zy.$$

Let

$G =$ the moment of the resultant couple, which, as shown in Article 171, may be represented by its axis;

$d =$ the angle which the axis of the resultant couple makes with the axis of x,

$e =$ angle which it makes with the axis of y,

$f =$ angle which it makes with the axis of z;

then will the component of the resultant couple, in reference to the axis of x, be

$$G \cos d ;$$

and similarly for the others. Hence, we have for equilibrium,

$$G \cos d = Zy - Yz,$$
$$G \cos e = Xz - Zx,$$
$$G \cos f = Yx - Xy.$$

If the *given* forces are in equilibrium in reference to rotation, we have

$$G = 0;$$
$$\therefore Zy = Yz,$$
$$Xz = Zx,$$
$$Yx = Xy;$$

the last of which may be deduced from the two preceding by eliminating z.

In many cases it is advisable to find the moments of the forces directly instead of the moments of the components.

Problems.

269. *A cord is secured at the points A and C, and sustains a weight attached to it at B; required the tension of each part of the cord, the weight of the cord being neglected, and the parts AB and BC being unequal.*

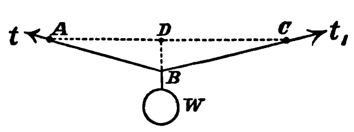


FIG. 126.

Let $t =$ the tension of AB,
$t_1 =$ the tension of BC, and
$W =$ the weight of the body.

Taking the origin of coördinates at any point as B, x horizontal, and y vertical, and resolving the forces horizontally and vertically, we have

$$X = t\cos BAD - t_1 \cos BCD = 0,$$
$$Y = t\sin BAD + t_1 \sin BCD - W = 0.$$

There being only two unknown quantities, the equation of moments will be unnecessary. If the origin of moments be at B, all the moments will vanish.

From the first of these equations we have

$$t\cos BAD = t_1 \cos BCD;$$

hence, the horizontal components of the tensions are equal to each other. From the last equation we have

$$t = \frac{\cos BCD}{\cos BAD} t_1;$$

which, combined with the second equation, gives

$$t = \frac{\sec BAD}{\tan BAD + \tan BCD} W;$$

$$t_1 = \frac{\sec BCD}{\tan BAD + \tan BCD} W.$$

270. *A weight W is attached to a cord DB of given length and pushed from a vertical wall by a strut BA of known length; required the tension, t, of the string, the compression, c, of the strut, and the upward push, F, on the vertical wall, when there is equilibrium.*

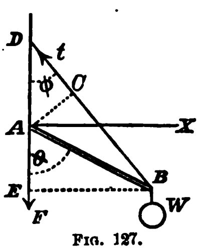


FIG. 127.

Let AD be known, and the angles of the triangle ABD computed.

Take the origin of coördinates at A, x horizontal, and y vertical, and positive upward. Resolving the forces acting at B, we have (see Article 157),

$$X = t\cos(90^\circ + \phi) + c\cos(270^\circ + \theta) + W\cos 270^\circ = 0;$$
$$Y = t\sin(90^\circ + \phi) + c\sin(270^\circ + \theta) + W\sin 270^\circ = 0;$$

and taking the moments of the forces directly, we have

$$\Sigma Fa = +t.AC + c.0 - W.BE = 0.$$

These reduced, give

$$-t\sin\phi + c\sin\theta = 0;$$
$$+t\cos\phi - c\cos\theta - W = 0;$$
$$t.AD\sin\phi - W.AB\sin\theta = 0.$$

These equations solved, observing that $\sin\phi = \frac{AC}{AD}$, and $\sin\theta = \frac{EB}{AB}$, give

$$t = \frac{AB\sin\theta}{AD\sin\phi}W = \frac{EB}{AC}W;$$

$$c = \frac{AB}{AD}W.$$

In a similar way, resolving the forces c, F, and X at A, and we find

$$F = c\cos\theta = \frac{AE}{AD}W;$$

and

$$X = c\sin\theta = \frac{EB}{AD}W.$$

271. *Two braces, or rafters, secured at their lower ends by a horizontal tie rod, carry a weight W at the point where they meet; required the vertical action at the supports, the tension of the horizontal tie, and the compression of the braces.*

First, consider the parts as rigid, and determine the relations between the *external* forces.

Let the braces be equally inclined, then will D, verti-

cally under W, be midway between A and B. Also, let V_1 be the vertical action of the support at A, and V that at B. Take the origin of coördinates at A, x horizontal and y vertical, and we have, from Fig. 128,

$$X = V_1 \cos 90^\circ + W \cos 270^\circ + V \cos 90^\circ = 0;$$
$$Y = V_1 \sin 90^\circ + W \sin 270^\circ + V \sin 90^\circ = 0;$$
$$\Sigma Fa = V_1 . 0 - W . AD + V . AB = 0;$$

and these give

$$X = 0;$$
$$Y = V_1 + V - W = 0;$$
$$\Sigma Fa = V . AB - \tfrac{1}{2} W . AB = 0;$$

which readily give

$$V = \tfrac{1}{2} W = V_1,$$

hence, each support sustains one-half the load W; a result readily deduced from the principle of the lever.

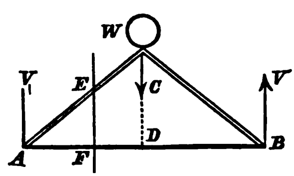


FIG. 128.

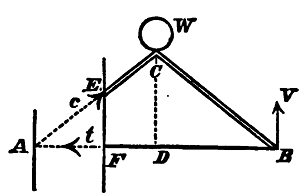


FIG. 129.

Secondly, to find the stresses which are transmitted along the pieces, conceive that a plane EF is passed through two of the pieces, cutting them at the points E and F. Let the part AEF, between the end A and the plane section, be removed, and let the forces c and t, equal to the compression and tension which previously existed, be substituted. The frame in the new condition will be in equilibrium, the forces c and t being the unknown

quantities to be determined from the equations of equilibrium. The direction of action of these forces is represented by the arrow heads, Fig. 129.

The origin of coördinates remaining at A, we have

$$X = V\cos 90^\circ + W\cos 270^\circ + t\cos 180^\circ \\ + c\cos CAD = 0,$$

$$Y = V\sin 90^\circ + W\sin 270^\circ + t\sin 180^\circ \\ + c\sin CAD = 0,$$

$$\Sigma Fa = V.AB - W.AD + t.0 + c.0 = 0;$$

which give, observing that $V = \frac{1}{2}W$,

$$-t + c\cos CAD = 0,$$
$$-\tfrac{1}{2}W + c\sin CAD = 0;$$

and these give

$$c = \tfrac{1}{2}W\operatorname{cosec} CAD,$$
$$t = \tfrac{1}{2}W\cot CAD;$$

or,

$$c\cos CAD = t,$$
$$c\sin CAD = \tfrac{1}{2}W;$$

from which it appears that the *horizontal component of the compression on AC equals the tension of the horizontal tie AB*, and

The vertical component of the compression on AC equals the vertical action at A.

In a similar way the stresses on any frame may be determined, provided the plane section does not intersect more than three acting members. There being three equations for equilibrium for forces in one plane, two for forces, and one for moments, only three unknown quantities can be determined from them.

272. Remark.—These solutions are given that the student may learn the *process* of establishing the general equations, and not because the solutions are the shortest

or the most simple. The solutions here given are ap ently longer than are necessary, for those quantities terms which reduce to zero need not be entered in equations; but it was thought best, for practice, to e every force in each equation. The Analytical meth not used because it makes the solution of a partic problem shorter than by special methods, but becau *establishes a uniform method of procedure*, and also *nishes a powerful method of investigation in more* (*cult problems.*

273. The preceding problem may be solved by following shorter method. Let Cb represent the weigh and ba be drawn parallel to and ad horizontal; then wil represent the compression CB, $aC = ab =$ the comp sion on CA, and ad the ten on AB. $Cd = db = \frac{1}{2}W$; he we have

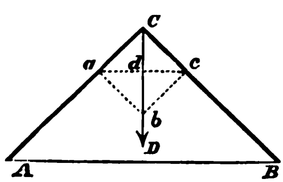


FIG. 130.

$$ad = Cd \cot Cad;$$

or,

$$t = \tfrac{1}{2} W \cot CAB;$$

and

$$aC = Cd \operatorname{cosec} Cad,$$

or,

$$c = \tfrac{1}{2} W \operatorname{cosec} CAB.$$

EXAMPLES.

1. In Fig. 126, if $W = 100$ lbs., $AC = 5$ ft., $AB =$ $= 3$ ft., what will be the tension of the cord?
2. In the same figure, if the length of the cord is 1(AC, 5 ft., what must be the length of AB that tension of it shall be twice that of BC?

3. If AB be horizontal, in Fig. 126, and BC inclined, find the tension on AB and BC.
4. In Fig. 128, if AB is 20 ft., DC 10 ft., and W 500 lbs., find the tension on AB and the compression on AC.
5. Required the inclination of the rafters, so that the stress on each will equal the weight W.
6. Required the inclination of the rafters so that the stress on AB shall equal the weight W.

EXERCISES.

1. In Fig. 122, if the resultant of F and F_1 equals R, and another force equal to R be introduced at the point of concurrence, and another force equal and opposite to R acting on another part of the body be also introduced, will the four forces have a single resultant?
2. If the force R, at the concurrence of F and F_1 be removed, will the three remaining forces have a single resultant?
3. In Fig. 125, if the point of application of the force be in the second angle, both components will turn the point of application right-handed; does the expression for the moments need to be modified for this case?
4. In Fig. 126, can the cord be drawn into a straight line when the weight B is upon it?
5. If the parts of the cord AB and BC are vertical, what will be the tension on each part?
6. If the points of support A and C are not in the same horizontal, while the inclination of the parts AB and BC remain the same, will the tension remain the same?
7. If the weight be moved to and fro on the cord, in what curve will be the locus of the point B?
8. In Fig. 128, if the lower ends of the braces, A and B, be moved towards each other, will the compression on the braces be increased or decreased? Will the thrust at the lower ends be changed?
9. What will be the effect upon the stresses if the apex be lowered by lengthening the cord AB?
10. If a vertical strut CD be placed under C, will there be any stress on the rafters?

CHAPTER XIII.

STRENGTH OF BARS AND BEAMS.

274. Strength of Prismatic Bars.—It has been served, Articles 129 and 130, that, if a solid be pulled the direction of its length, it will be elongated. We know that if the pulling force be sufficiently great, bar will be broken.

The strength of solids is determined by experim The most common way of doing it is to take a bar wl

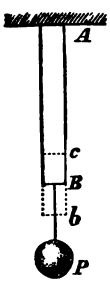


FIG. 131.

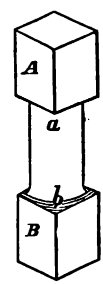


FIG. 132.

cross-section is somewhat larger than the section whic is desired to test, and then turn down a portion near middle of the piece to an exact cylinder of definite meter. The piece is then placed in a powerful macl and pulled asunder. The total pulling force being r sured, the stress per unit of section is easily compu By experimenting upon pieces of different diameters lengths, and with different materials, certain *general f* or *laws* have been determined.

. **Results.**—*The strength of prismatic bars varies ly as their cross-section.* This is the law assumed ctice, and it appears to be nearly evident without iment; but, like all other physical laws pertaining ds, *it is not rigidly exact.* It is found that, when duced portion *ab* is very short, the piece is stronger when it is longer, but no law for this increase of th is known. Bars of small cross-section generally r to be stronger in proportion than those of larger section, but the difference may be more apparent than or it may be due to the difficulty of producing a pull y in line with the axis of the larger piece, and cross-s will generally break a piece more easily than a pull. If, however, the pieces are made small by so forged before they are turned, instead of being d down from larger pieces, the difference in strength be due to the process of manufacture.

ien a piece is elongated, its cross-section is contracted. the material is ductile, the contraction may be con-ble before fracture takes place. Thus, in some ex-ents upon wrought iron, the cross-section has been ed in this way to one-half of its original area. the part *ab*, Fig. 132, is *very short*, the contraction ch less than when it is longer. The shoulders of the portion appear to aid in resisting the contraction smaller part, and this may account for the increase ength above referred to.

. **Modulus of Tenacity.**—*The pull, in pounds, will break a bar whose cross-section is one square is a measure of the tenacity of the material* and is THE MODULUS OF TENACITY. It is represented

Mean Values of T.

For wrought iron.........	45,000 lbs.	to	60,000 lbs.
For steel................	70,000 "	to	190,000 "
American cast-iron.......	20,000 "	to	45,000 "
English cast-iron........	13,000 "	to	25,000 "
Ash (*English*)...........	15,000 "	to	17,000 "
Oak (*English*)...........	9,000 "	to	15,000 "

277. Formula for the Tenacity of Solids.

Let

T = the modulus of Tenacity,
P = the pulling force,
S = the area of the cross-section;

then we have

$$P = TS.$$

278. Strength of Beams.—In order to make a formula for the strength of beams, it is necessary to know the law of action of the forces within the beam. Within the elastic limits this law is quite perfectly known, but, when a state bordering on rupture is reached, the law of action is quite complex, and is not accurately known. But the same *law of resistance* is assumed for both cases, and the error, if any, which results from the assumption is corrected by means of a constant factor.

279. Law of Resistance.—When a beam is fixed at one end and loaded at the free end, as in Fig. 133, the fibres on the upper side are elongated and on the lower side, compressed, and there is a surface between the extended and compressed elements which is neutral, and is called a *neutral surface.* It is assumed that the stress on the fibres varies directly as their distance from the

neutral surface. The intersection of the neutral surface with a vertical, longitudinal plane is called *the neutral axis.*

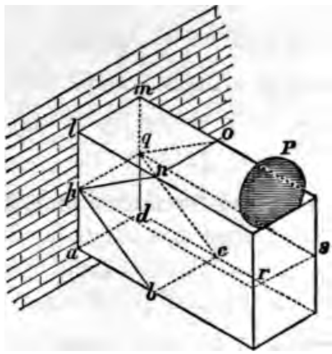


FIG. 133.

280. Modulus of Rupture.—*The Modulus of Rupture is the stress at the instant of rupture on a unit of area of the cross-section which is most remote from the neutral axis.* It is represented by R.

281. Expression for the Moment of Resistance of Rectangular Beams.—In Fig. 133, let pr be the neutral axis. Take $ln = mo = ab = dc = R$. Pass the planes po and pc, then will the wedge pq-$noml$ represent the total pulling stress, and pq-$abcd$ the total pushing or compressive stress.

Let

$$b = lm = \text{the breadth of the beam, and}$$
$$d = la = \text{its depth; then}$$
$$\tfrac{1}{2}bd = \text{area } plmq, \text{ and}$$
$$\tfrac{1}{4}Rbd = \text{the volume of the upper or lower wedge.}$$

The moment of this resistance equals the total resistance into the distance of the centre of gravity of the wedge from the neutral surface. The centre of gravity of the

wedge is at the same distance from the neutral surface as that of the triangle pnl; that is $\frac{2}{3}pl$; hence, the arm of the resistance is $\frac{2}{3}pl = \frac{2}{3} \times \frac{1}{2}d = \frac{1}{3}d$ and the moment is

$$\tfrac{1}{4}Rbd \,.\, \tfrac{1}{3}d = \tfrac{1}{12}Rbd^2;$$

and the total moment of resistance of both the upper and lower stresses will be twice this value, or

$$\tfrac{1}{6}Rbd^2.$$

This expression is true for all rectangular beams whose sides are vertical, whether the beam be fixed at one end, supported at its ends, or otherwise, and whether the beam be loaded at one point, several points, or uniformly loaded.

282. Problems.—1. *Let a rectangular beam be fixed at one end, to find a load P, placed on the free end, which will just break the beam.*

In Fig. 133, let $l = pr =$ the length of the beam; then the moment of P, in reference to the fixed end as the origin of moments will be

$$Pl;$$

and this must equal the moment of stress; hence,

$$Pl = \tfrac{1}{6}Rbd^2$$

$$\therefore P = R\frac{bd^2}{6l} \quad . \quad . \quad . \quad . \quad (1)$$

From this equation we find

$$R = \frac{6Pl}{bd^2}; \quad . \quad . \quad . \quad (2)$$

from which the value of R may be easily computed when the breaking weight P and the dimensions of the beam are known. The value of R has been determined for a variety of substances, and the results given in tables in

works on the Resistance of Materials. It is a remarkable fact that the value of R is not the same as T for any substance.

Equation (1) is true for any stress less than that which will break the beam. It is customary, in practice, to use a fractional part of R, called the *factor of safety*, when beams are to be proportioned. About $\frac{1}{5}$ to $\frac{1}{6}$ of R is used for wrought iron, $\frac{1}{10}$ for wood, and $\frac{1}{6}$ to $\frac{1}{8}$ for cast iron, which values give for the *safe* value of R about

10,000 lbs. to 12,000 lbs. for wrought iron,
1,000 lbs. to 1,200 lbs. for wood,
5,000 lbs. to 6,000 lbs. for cast iron, and
12,000 lbs. to 20,000 lbs. for steel.

Any one of the quantities of equation (1) may be found when all the others are given.

2. *Let the beam be fixed at one end and uniformly loaded.*

Let W equal the total load; then will the lever-arm of the load be the distance from the fixed end B to the centre of gravity of the load, which is $\frac{1}{2}l$; hence, the moment is

$$\tfrac{1}{2}Wl,$$

and the equation for equilibrium becomes

$$\tfrac{1}{2}Wl = \tfrac{1}{6}Rbd^2.$$

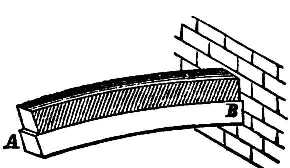


FIG. 134.

3. *Let the beam be supported at its ends and loaded in the middle.*

The point where a beam is most liable to break is called *the dangerous section.* If the beam at this section is strong enough to support the load the beam will not break. In this problem the dangerous section is at the middle of

the length; hence, the equation of moments should be established for this point.

The reaction of each support will be $\frac{1}{2}P$, and the arm of this force, in reference to the middle point of the beam as an origin of moments, is $\frac{1}{2}l$, and the moment is

$$\tfrac{1}{2}P.\tfrac{1}{2}l,$$

and the equation becomes

$$\tfrac{1}{4}Pl = \tfrac{1}{6}Rbd^2.$$

4. *Let the beam be supported at its ends and uniformly loaded.*

Each support will sustain one-half the load, or

$$\tfrac{1}{2}W,$$

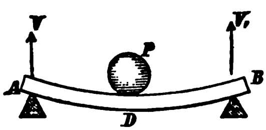


FIG. 135.

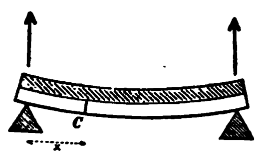


FIG. 136.

which acts upward and tends to turn the beam one way. The arm of this force is $\frac{1}{2}l$ in reference to the centre of the beam taken as the origin of moments, and its moment will be

$$\tfrac{1}{2}W.\tfrac{1}{2}l = \tfrac{1}{4}Wl.$$

The load between the middle of the beam and one of the supports, which is also $\frac{1}{2}W$, acts downward, tending to turn the beam in the opposite direction to that of the former force; and the arm of this force is the distance from the origin of moments to the centre of gravity of this load, which is $\frac{1}{4}l$; hence its moment is $-\frac{1}{2}W.\frac{1}{4}l$; and the resultant moment will be

$$\tfrac{1}{4}Wl - \tfrac{1}{8}Wl = \tfrac{1}{8}Wl;$$

and the equation for equilibrium becomes

$$\tfrac{1}{8}Wl = \tfrac{1}{6}Rbd^2.$$

EXAMPLES.

[It is best to reduce all the linear dimensions to inches.]

A rectangular beam, whose depth is 8 inches, length 8 feet, $R = 1{,}400$ lbs., is supported at its ends; required the breadth so that it will carry 500 lbs. per foot of its length. *Ans.* $b = 3\frac{3}{14}$ *inches.*

A rectangular beam is fixed at one end and is required to carry 1,000 lbs. at the free end; if $l = 8$ feet, $R = 1{,}200$ lbs., and the depth is four times the breadth, required the breadth and depth.

A rectangular beam, whose length is 12 feet, breadth 2 inches, modulus of rupture 10,000 pounds, is supported at its ends; required the depth so that it will support 8,000 pounds placed at the middle of the beam.

If a beam whose length is 10 feet, breadth 4 inches, and depth 9 inches, is supported at its ends, and broken by a weight of 20,000 pounds placed at the middle of its length, what is the value of R?

A beam whose breadth is $1\frac{1}{2}$ inches, depth $3\frac{1}{2}$ inches, is supported at its ends and loaded at the middle with 10,000 pounds; what must be its length so that the stress on the outermost fibres shall be 20,000 pounds?

A beam whose length is 15 feet, breadth 6 inches, and depth 12 inches, is supported at its ends; required the load, uniformly distributed, which it will safely support, calling the safe value of R 12,000 pounds.

An iron beam, two inches square, projects horizontally from a wall; what must be its length to break itself at the wall, the value of R being 30,000 pounds, and the weight of the material being $\frac{1}{4}$ pound per cubic inch?

CHAPTER XIV.

MOTION OF A PARTICLE ON AN INCLINED PLANE.

283. *To find the formulas for the movement of a particle down an inclined plane by the action of gravity, all resistances being neglected.*

Let

AC be the plane,

$W =$ the weight of the particle,

$F =$ the component of the force of gravity parallel to the plane,

$s =$ the distance over which the body moves in the time t,

$v =$ the velocity acquired in the time t, and

$\phi = CAB =$ the angle of elevation of the plane.

The force of gravity resolved parallel to the plane gives

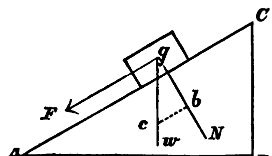


FIG. 131.

$$F = W \sin \phi,$$

which, being constant, will, according to Article 60, produce a constant acceleration, the value of which may be found by Article 86, and is

$$f = \frac{F}{M} = \frac{W \sin \phi}{\frac{W}{g}} = g \sin \phi.$$

This value of f, in the equations of Article 24, gives

$$v = gt \sin \phi, \quad . \quad . \quad . \quad (1)$$

$$v = \sqrt{2gs \sin \phi}, \quad . \quad . \quad . \quad (2)$$

$$s = \tfrac{1}{2} gt^2 \sin \phi, \quad . \quad . \quad . \quad (3)$$

$$t = \sqrt{\frac{2s}{g \sin \phi}}. \quad . \quad . \quad . \quad (4)$$

Let $s = AC$, then, from the figure, we have

$$s \sin \phi = BC,$$

which, substituted in equation (2), gives

$$v = \sqrt{2g \,.\, BC},$$

which equals the velocity acquired in falling through a vertical height BC. Since a similar result will follow from a plane having any inclination, and hence, from several successive planes having different inclinations, it follows that *the velocity acquired by a particle in falling from one point to another when there are no resistances, will be independent of the path over which it moves, and will equal the velocity acquired in falling vertically through a height equal to the height of one point above the other.*

This result also follows directly from the principles of Kinetic Energy, for the energy imparted by gravity equals the weight of the body into the height through which it acts.

The formulas of this article may be deduced from those of Article 247, by supposing that one of the bodies vanishes.

284. Initial Velocity.—If the body is projected down the plane, or, in other words, has an initial velocity v_0 at the instant t begins to be reckoned, we have,

$$v = v_0 + gt \sin \phi,$$

$$s = v_0 t + \tfrac{1}{2} gt^2 \sin \phi;$$

$$v^2 = v_0^2 + 2gs \sin \phi,$$

and, if the body be projected up the plane with a velocity v_0, we have

$$v = v_0 - gt \sin \phi,$$
$$s = v_0 t - \tfrac{1}{2} g t^2 \sin \phi.$$
$$v^2 = v_0^2 - 2gs \sin \phi,$$

Problems.

285. *The times of descent down all chords of a vertical circle, which pass through either extremity of a vertical diameter, are the same.*

Let ACB be the circle, and AC any chord drawn through A. According to equation (4) of Article 283, the time of descent down AC will be

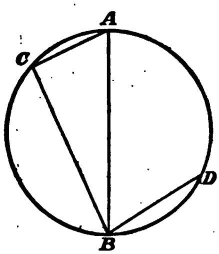


FIG. 132.

$$t = \sqrt{\frac{2AC}{g \sin \phi}}.$$

Draw CB, then will the angle ACB be right, for it is inscribed in a semicircle. The angle ϕ is the complement of CAB; hence,

$$\sin \phi = \cos A = \frac{AC}{AB},$$

which, substituted in the preceding equation, gives

$$t = \sqrt{\frac{2AB}{g}},$$

which is not only constant, but is the time of falling vertically through the diameter AB.

286. *Find the straight line from a given point to a given inclined plane, down which a body will descend in the least time.*

Let A be the point and BC the plane. Through A draw a vertical line AO, and on it take O at such a point that a circle described with the radius OA shall be tangent to BC; then will the chord AD, drawn from the upper extremity A of the diameter, to the point of tangency D, be the required line. For, all other lines drawn from A to the right line will be secants of the circle, and, according to the preceding article, the times of descent down

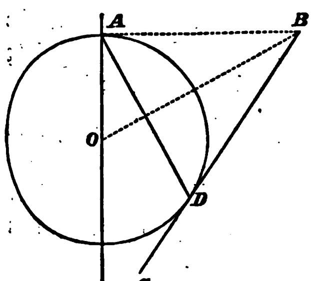


Fig. 133.

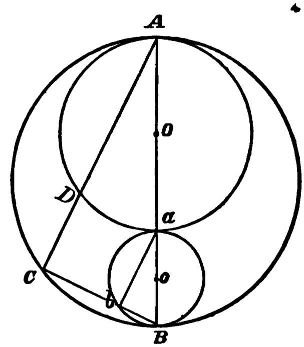


Fig. 134.

the internal portions will be the same, and hence, the time of descent down the whole line will be greater than that down AD.

[To find the centre of the circle, draw a horizontal line AB, which line will be tangent to the circle, and bisect the angle B by the line OB; the point of intersection, O, of the lines AO and BO, will be the centre required. From the property of tangents, BA will equal BD.]

287. *To find the straight line of quickest descent from one circle to another, the centres of the circles being in the same vertical line, and one tangent to the other internally.*

Let A be the point of tangency. Draw any secant ADC,

and draw BC. On aB as a diameter, draw the circle abB, and draw the chord ab. The angles C and b will be right, for each is inscribed in a semicircle, and ab will be parallel to DC. The time of descent from D to C will be the same as from a to b, which is the same as from a to B; hence, the time of descent from any point of the circle whose centre is O, down the external portion of the secant drawn from A through a point D to the external circle, is constant and equal to that down the vertical aB. If D is a given point on the circle ADa, then DC will be the line of quickest descent. If the common tangent is at the lowest point of the circles, at B, and C the given point; draw the secant BC, and Cb will be the required line.

288. *To find the straight line of quickest descent from a given circle to a given line.*

Let AEB be the circle, and CD the line. Through A draw the horizontal line AC, make CD equal to AC, and draw AD; the line required will be ED.

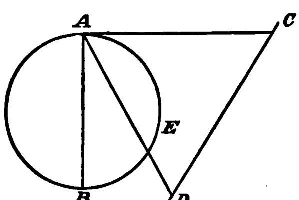


Fig. 135.

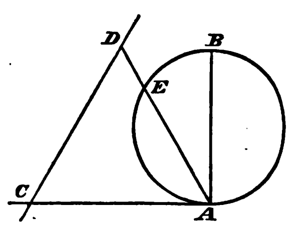


Fig. 136.

289. *To find the line of quickest descent from a straight line to a given circle.*

Let CD be the line and AEB the circle. Draw the horizontal tangent AC; make $CD = AC$; draw DA, and DE will be the required line.

290. *Find the time of descent down a rough inclined plane.*

Let the forces be resolved as in Fig. 131, and μ be the coefficient of friction. The normal pressure will be

$$N = W \cos\phi,$$

and the resistance due to friction will be

$$\mu W \cos\phi;$$

hence, the effective moving force will be

$$W \sin\phi - \mu W \cos\phi$$
$$= (\sin\phi - \mu\cos\phi) W;$$

and the constant acceleration will be

$$f = (\sin\phi - \mu\cos\phi) g;$$

which, substituted in the equations of Article 24, gives

$$v = (\sin\phi - \mu\cos\phi) gt, \quad . \quad . \quad . \quad . \quad (1)$$

$$v = \sqrt{2(\sin\phi - \mu\cos\phi) gs}, \quad . \quad . \quad . \quad (2)$$

$$s = \tfrac{1}{2}(\sin\phi - \mu\cos\phi) gt^2. \quad . \quad . \quad . \quad (3)$$

291. Approximate Formulas.—Substitute the values of the sine and cosine in the preceding value of f, and we have

$$f = \left(\frac{BC}{AC} - \mu\frac{AB}{AC}\right) g;$$

but, when the angle A is small, AB may be considered as equal to AC, and $BC \div AC$ will be the *grade* (a term which is in common use in Railroad Engineering). Let γ be the grade, and the formula becomes

$$f = (\gamma - \mu) g;$$

and we have

$$v = (\gamma - \mu) gt, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$v = \sqrt{2(\gamma - \mu) gs}, \quad . \quad . \quad . \quad . \quad (2)$$

$$s = \tfrac{1}{2}(\gamma - \mu) gt^2. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

292. Formulas adapted *to the movement of cars on planes of small inclination.*

The grade is commonly given in feet per mile. Let h be the elevation per mile, then

$$\gamma = \frac{h}{5280},$$

and the friction may be taken at 7.5 lbs. per ton gross; hence,

$$\mu = \frac{7.5}{2240} = \frac{1}{300};$$

and the formulas become, with sufficient accuracy,

$$v = \frac{1}{164}(h - 17.6)t, \quad . \quad . \quad . \quad (1)$$

$$v = \frac{1}{9}\sqrt{(h - 17.6)s}, \quad . \quad . \quad . \quad (2)$$

$$s = \frac{1}{328}(h - 17.6)t^2. \quad . \quad . \quad (3)$$

If the body has a velocity, the distance it will move on a horizontal plane, in being brought to rest, may be found by making $h = o$, in the second of these equations, and the time of the movement, from the first, after making $h = o$ in that equation; hence we will have for this case

$$t = 9\tfrac{1}{3}v \text{ (nearly)}, \quad . \quad . \quad . \quad . \quad (4)$$
$$s = 4.6v^2 \text{ (very nearly)}. \quad . \quad . \quad (5)$$

EXAMPLES.

1. A smooth inclined plane is 100 feet long; what must be its inclination that the time of descent of a parti cle down it shall be 5 seconds?

2. A body is projected up a smooth plane whose slope (that is its inclination with the horizontal) is 45 degrees, with a velocity of 50 feet per second; find its position at the end of 3 seconds; five seconds; ten seconds.

3. A body starts from rest at the top of an inclined plane whose length is 100 feet, and height 20 feet; with what velocity must a body be projected up the plane from its foot in order to meet the former one at the middle of the plane?

4. Find the line of quickest descent from a point without a circle to the circumference of the circle.

5. A train of cars starts from rest on an inclined plane and runs down it by the force of gravity only; the grade being 40 feet to the mile and the coefficient of friction $\frac{1}{300}$, what will be its velocity at the end of one mile?

6. A car starts from the upper end of an inclined plane whose length is one-half of a mile, and grade 50 feet per mile, coefficient of friction $\frac{1}{300}$, and runs down the plane; how far will it go on a succeeding horizontal plane before being brought to rest, the coefficient of friction remaining constant?

7. In the preceding example, required the time of the movement down the plane; also the time on the horizontal plane.

8. Required the height of a plane which is 1,200 feet long, so that a car starting at rest from the upper end of it, and running down shall go just 1,000 feet on a succeeding horizontal plane, the coefficient of friction being $\frac{1}{300}$.

Suppose that there is an available height of 25 feet for

an inclined plane, required its length so that a body descending it shall go just 800 feet on a succeeding horizontal plane, the coefficient of friction being $\frac{1}{300}$.

[The examples from 5 to 9 inclusive may be solved by the formulas of the last article. The resistance of the air, and the effect of the rolling mass in the wheels, is neglected. In the formulas of the preceding article, t is in seconds, v in feet per second, and s in feet.]

CHAPTER XV.

PROJECTILES.

293. A Projectile is a body thrown into space and ıcted upon by gravity. It is important in the art of Gunıery; but the solution of problems in that Art pertainng to the flight of the projectile, to be of any value, involves the resistance of the air, and this element so comılicates the problem that its solution will not be attempted n this work. Some interesting properties may, however, e easily determined by assuming that the body moves in ı vacuum; and it will be found that the formulas thus leduced are of practical value in the science of Hydrolynamics.

294. The path *of a projectile* is the line which it decribes. This assumes that the body is a particle, but if he size be considered, we would say that it is the line decribed by the centre of its mass.

The horizontal range *of a projectile* is the distance lB, Fig. 139, from the point A, where the body is proıcted, to the point B, where the path crosses the horizonıl plane passing through A.

295. The path described by a projectile in a vacum is a parabola.

Let A be the point from which the projection is made, lB the line of projection, and v the velocity of projection. n a certain time, t, the body will have moved a certain istance, AB, under the action of the impulse only; ence,

$$AB = vt.$$

In the same time it would have moved a distance $AD = BC$ under the action of gravity only; or

$$BC = \tfrac{1}{2}gt^2.$$

When both motions are simultaneous, it will be at the point C, at the end of the time t. Eliminating t from these equations, gives

$$AB^2 = \frac{2v^2}{g} AD.$$

Let h = the height through which a body must fall in

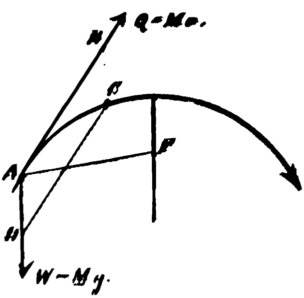


Fig. 137.

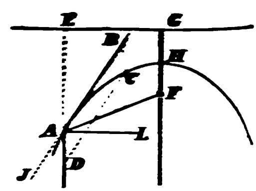


Fig. 138.

vacuo to acquire a velocity v; then, according to Article 72,

$$v^2 = 2gh,$$

which, substituted in the preceding equation, gives

$$AB^2 = 4h \, . \, AD;$$

which is the equation of a parabola. The line AB is a tangent to the curve, and AD is a diameter.

200. To find the velocity at any point *of the path.*

Let A, Fig. 138, be the point. The velocity will be the same as if the body had fallen a height h; but $4h$,

according to the last equation, is the parameter of the parabola to the diameter AD. Let EG be the directrix of the parabola, then will the velocity be equal to that due to a fall through a vertical height EA; or,

$$v = \sqrt{2g \cdot EA}.$$

297. To find the position of the focus. Let F be the focus, and AL be drawn horizontally. According to a property of the parabola, the tangent AB makes equal angles with the diameter AD and the line AF drawn from the focus to the point of tangency; hence,

$$\begin{aligned} BAF &= JAD \\ &= EAB = 90^\circ - BAL; \end{aligned}$$

therefore the angle between the tangent to the path, and the line from the point of tangency to the focus, is the complement of the angle of elevation of projection.

The distance $AF = EA$, is, according to the preceding article,

$$EA = \frac{v^2}{2g};$$

hence, when the *velocity of projection and the angle of elevation are given, the focus can be found.*

298. To find the equation of the path *when referred to rectangular axes.*

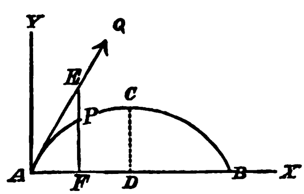


FIG. 139.

Let A be the point of projection, and P the position of the body at the end of a time t. Take the origin of coördinates at A, x horizontal, and y vertical. Let $\alpha = EAF$. The coördinates of P being

$$AF = x, \quad FP = y,$$

we have,

$$
\begin{aligned}
&AE = vt,\\
&AF = vt\cos a = x, \quad . \quad . \quad . \quad . \quad . \quad (1)\\
&EF = vt\sin a,\\
&EP = \tfrac{1}{2}gt^2,\\
\therefore\ &FP = vt\sin a - \tfrac{1}{2}gt^2 = y \quad . \quad . \quad . \quad (2)
\end{aligned}
$$

Eliminating t from equations (1) and (2) gives

$$
\begin{aligned}
y &= x\tan a - \frac{gx^2}{2v^2\cos^2 a}\\
&= x\tan a - \frac{x^2}{4h\cos^2 a}. \quad . \quad . \quad . \quad . \quad (3)
\end{aligned}
$$

which is the required equation.

299. To find the range AB. Making $y = 0$ in equation (3), we have

$$x = 0,$$

and

$$x = 4h\cos a\sin a = 2h\sin 2a = AB.$$

300. To find the Time of Flight. Substitute the value of AB, given in the preceding article, in equation (1) of Article 298, and we find

$$t = \frac{4h\sin a}{v} = 2\frac{v\sin a}{g}.$$

301. To find the greatest height in the path.—It will be vertically over D, the middle point of AB. From Article 299 we find

$$AD = 2h\cos a\sin a;$$

which, substituted for x in equation (3) of Article 298, gives

$$CD = h\sin^2 a.$$

302. Greatest Range.—*To find the angle of elevation 'the projectile which will give the greatest range.*
This requires that the value of AB, in Article 299, all be a maximum; hence,

$$\sin 2a = 1$$
$$\therefore a = 45°.$$

303. Greatest Height.—*To find the angle of elevation ' projection which shall give the greatest height.* This quires that the value of CD, Article 301, shall be a aximum; hence,

$$\sin^2 a = 1$$
$$\therefore a = 90°;$$

ıat is, the projection must be vertical.

304. Equal Ranges.—Since the sine of an angle equals

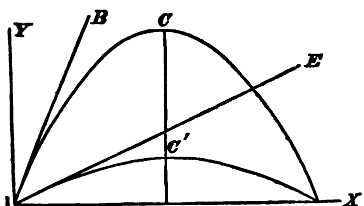


FIG. 140.

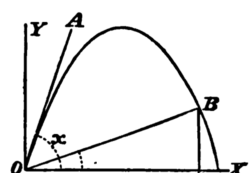


FIG. 141.

ıe sine of its supplement, the value of AB, Article 299, ʼill be the same

$$\text{for } \sin 2a, \text{ as for } \sin(180° - 2a);$$

ence, the range will be the same for the angles

$$a, \text{ and } 90° - a;$$

hat is, if the angles of elevation of two projectiles be the omplements of each other, and have the same initial elocity, their ranges will be equal to each other.

305. Range on an Inclined Plane.—*If an inclined plane, OB, passes through the point of projection, it is required to find the range on the plane.*

Let the angle $BOC = \phi$, then will

$$CB = y = x \tan \phi,$$

and this value of y, substituted in equation (3) of Article 298, gives

$$x = OC = \frac{2v^2 \cos \alpha \sin (\alpha - \phi)}{g \cos \phi};$$

therefore,

$$OB = OC \sec \phi = \frac{2v^2 \cos \alpha \sin (\alpha - \phi)}{g \cos^2 \phi}.$$

When the projectile is moving in a vacuum, gravity is the only force which *acts* upon it.

306. Problems.—1. *Find the equation of the path when the body is projected horizontally.*

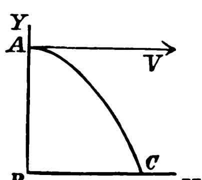


FIG. 142.

In this case $\alpha = 0$ in equation (3) of Article 298; hence we have

$$x^2 = -4hy.$$

If y be taken positive downwards, we have

$$x^2 = 4hy.$$

2. *When the body is projected horizontally, find the range BC on a horizontal plane below the point of projection, and the time of flight.*

Let $AB = y_1$. Substitute y_1 for y in the preceding equation, and solve for x, and we find

$$BC = x = 2\sqrt{hy_1}.$$

The time of flight will be that necessary to move the

horizontal distance BC with the velocity v. Let t be the time, then

$$vt = BC = 2\sqrt{hy_1}$$

$$\therefore t = \frac{2\sqrt{hy_1}}{v}.$$

3. *To find the velocity and angle of elevation so that a projectile shall pass through two points whose coördinates are known.*

Let x_1 and y_1 be the coördinates of one point, x_2 and y_2 the coördinates of the other, and these substituted for x and y respectively in equation (3) of Article 298, give

$$y_1 = x_1 \tan a - \frac{x_1^2}{4h \cos^2 a};$$

$$y_2 = x_2 \tan a - \frac{x_2^2}{4h \cos^2 a}.$$

Eliminating h gives

$$\tan a = \frac{y_1 x_2^2 - y_2 x_1^2}{x_1 x_2 (x_2 - x_1)}.$$

Eliminating $\tan a$, gives

$$h = \frac{x_1 x_2 (x_2 - x_1)}{4 \cos^2 a (y_1 x_2 - y_2 x_1)};$$

hence,

$$v = \sqrt{2gh} = \frac{1}{2 \cos a} \sqrt{\frac{2g x_1 x_2 (x_2 - x_1)}{y_1 x_2 - y_2 x_1}}.$$

EXAMPLES.

1. A body is projected with a velocity of $5g$ at an angle of elevation of 45° to the horizon; determine the range and greatest height.

2. A body is projected horizontally with a velocity equal to that acquired by a body falling through a height of 15 feet; required the range BC, Fig. 142, on a plane 12 feet below the point of projection.

3. The horizontal range of a projectile is 1,000 feet, and time of flight 15 seconds; required the angle of elevation, velocity of projection, and greatest altitude.

$$\textit{Ans.}\ a = 74^\circ\ 33'\ 9''.$$
$$v = 250.29 \text{ feet.}$$
$$A = 904.69 \text{ feet.}$$

4. A body projected at an angle of elevation of 45° has a horizontal range of 25,000 feet; required the velocity of projection, the greatest altitude, and time of flight.

$$\textit{Ans.}\ v = 896 \text{ feet.}$$
$$A = 6{,}250 \text{ feet.}$$
$$t = 39 + \text{ seconds.}$$

5. The horizontal range of a projectile is four times its greatest height; required the angle of elevation.

6. A body is projected from the top of a tower 150 feet high, at an angle of elevation of 45°, with a velocity of 75 feet per second; required the range on the horizontal plane passing through its foot.

7. The eaves of a house are 25 feet from the ground, and the roof is inclined at an angle of 30° to the horizon, the ridge being 32 feet from the ground; find where a smooth sphere sliding down the roof will strike the ground.

8. A body passes through two points whose coördinates are $x_1 = 400$ feet, $y_1 = 60$ feet, $x_2 = 600$ feet, and $y_2 = 50$ feet; find the velocity, angle of elevation, horizontal range, and time of flight.

The effect of the resistance of the air may be illustrated by the fact that a certain ball being projected with a velocity of 1,000 feet per second in the air, its range was found to be about 5,000 feet, instead of 31,250 feet, as it would have been in a vacuum.

EXERCISES.

A body is projected vertically upwards; what will be its range?

A ship is sailing due east at the rate of 5 miles per hour; if two bodies are projected from the ship at the same angle of elevation, one in an easterly direction and the other in a westerly direction, will the range of the projectiles be the same?

If a ship sails at the rate of 15 miles per hour, and a body be thrown from it in an opposite direction with a velocity of 11 feet per second, required the actual velocity of projection.

Considering the rotation of the earth, will the range of a projectile be the same if it be fired in a westerly direction that it will if fired in an easterly direction?

If a projectile be struck horizontally, when at its highest point, by another body, will it reach the horizontal plane in the same time as if it had not been struck?

If two particles be projected from the same point with the same velocity, but the angle of elevation of one is as much above 45° as the other is below it, will their ranges on a horizontal plane be the same?

If two bodies are projected from the same point on a smooth horizontal plane, with different velocities, and in different directions, show that the straight line which joins them always moves parallel to itself.

CHAPTER XVI.

CENTRAL FORCES.

307. A Central Force is one which acts directly towards or from a point. The point is called the *centre* of the force. The body may move directly towards or from the centre, or it may move about it in a curved path, called the *orbit*. The latter is the one commonly understood when central forces are referred to. Thus, the centre of the sun is considered as the centre of the attractive force which acts upon the planets, and the planets move in *orbits* about the sun. Similarly, the centre of the earth is considered as the centre of the force which acts upon the moon, and causes it to move in its orbit. When a stone is whirled in a sling, the centre of the force which continually pulls upon the body is at the point where the string is held by the hand.

[In the case of the sun and planets, the centre about which they revolve is not the centre of the sun, but a point which is at the common centre of the system, and which is relatively near the centre of the sun.]

308. Centripetal Force is a name given to that central force which acts *towards* the centre. It is either attractive, or of the nature of a pull. Thus, the attractive force of the sun upon the planets, and of the primary planets upon their secondaries, is centripetal. The *pull* of the string upon the body whirled in a sling is centripetal. The cohesive force of the metal in a fly-wheel, which

prevents the rim from flying away from the centre, is of the same nature.

309. When a body moves in an orbit, *the centripetal force* constantly draws the body away from the straight line in which it tends to move. According to the FIRST LAW of motion, the body would move in a straight line if it were not acted upon by any force, but by the constant action of the central force, it moves in a curved line. The straight line in which the body tends to move, is a tangent to the orbit, as DG, Fig. 143.

310. The Orbit.—If a body at rest be left to the action of a central force, it will move directly towards or from that centre. The same will be true if it be *projected* directly towards or from the centre. In such cases the orbit is a straight line passing through the centre of the force.

Thus, if there were a hole through the earth so that a body could move from surface to surface, a body placed in the hole, or thrown directly towards or from the centre of the earth, would move under the action of a central force, and the path would be a straight line.

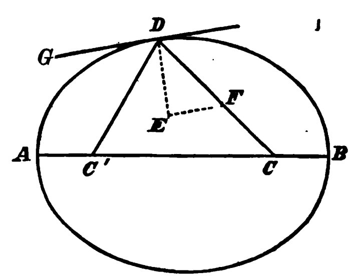


FIG. 143.

But if the body be projected at an angle with the line joining the body and the centre of force, it will move in a curved path. The orbits of all the planets are ellipses, with the sun at one of the foci. Thus, if ADB be an ellipse, representing an orbit of a

planet, C and C' the foci, then will the sun be at one of the latter points in reference to the orbit. It is generally assumed that comets move in parabolic orbits, the sun being at the focus. This is done so as to simplify the computations for determining their positions.

311. To find the orbit it is necessary to know the law of action of the central force, and the position, velocity, and direction of motion of the body at some point in the orbit. In the solar system the force varies inversely as the square of the distance from the centre. See Article 65. The complete investigation of this subject properly belongs to higher mathematics, and we shall consider, in this chapter, only that case in which the orbit is a circle.

312. Centrifugal Force.—This is a name given to a force which is equal and opposite to that which deflects the body from a tangent to the curve. If the body moves in the arc of a circle it will be equal and opposite to the centripetal force. If the orbit is not a circle the central force will act at an oblique angle to the tangent at nearly every point of the path. In Fig. 143, let the body be at D, and the centre of the force at C. Let DF represent the magnitude of the force, and let it be resolved into two forces, one, EF, parallel to the tangent, the other, DE, perpendicular to the tangent. If the body is moving in the direction DG, the component EF will retard the motion; if in the opposite direction, the motion will be accelerated; but in either case the component DE pulls the body *directly* away from the tangent. Then will the centrifugal force be equal and opposite to the normal component DE.

[The idea is often conveyed that the centrifugal and deflecting forces act upon the moving body at the same time, but such is

not the case; for, if they did, they would exactly neutralize each other, and the body would move in a straight line. Thus, when a train of cars runs around a curve, the forces under which the body moves are the propelling force of the steam and the deflecting force of the track. The *deflecting* force is *centripetal*. The so-called *centrifugal force* is exactly equal and opposite, and is a measure, in the contrary direction, of the pressure exerted by the rails of the track. If we consider the deflecting force as acting *between* the rail and wheel, then will the action of the force upon the wheel be *centripetal*, and the reaction, or pressure outward against the rail, be *centrifugal*. Some have also stated that the centrifugal force is that which causes the body to fly away, or to tend to fly away, from the centre, as if there were an active force radially outward, due to the revolution. But there really is no such force in this case. The body tends to go in a straight line, tangent to its path. Cut the string of a sling, or let go of one string, and the body starts off in a straight line. The term *centrifugal* is not so objectionable as the idea which it has been made to represent.]

Motion in the Circumference of a Circle.

313. To find the value of the central force *when e body describes the arc of a circle.* Conceive that the ›dy is held by a string fastened to a fixed point at the ›ntre of the circle; it is proposed to find the tension of ıe string as the body moves along the circumference. o solve this problem, first consider the conditions by hich a body may be made to move over the successive des of an inscribed regular polygon with a uniform elocity. Let $ABCD$, etc., be a regular inscribed polyon. If a particle moves over the side AB with a unirm velocity, it is required to determine the direction d magnitude of an impulse applied at B, that shall use the body to move along the side BC with the same locity that it did along AB.

Since the velocity is to be uniform, the sides AB and

BC will be proportional to the velocities along those sides; and the velocity along the chord BC will be the resultant of that along AB, and of that produced by the required impulse. On AB and BC, as sides, construct the rhombus $ABCF$; then will the diagonal BF be proportional to the velocity which must be produced by the required impulse, and will also represent the direction in which the impulse must act. See Articles 14 and 15.

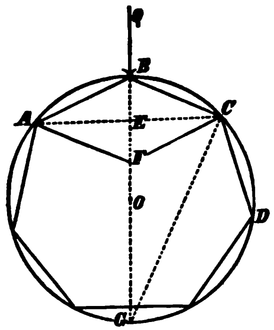


FIG. 144.

Let

$l = AB = BC =$ the length of a side of the polygon;

$s = BF$; $r = OB =$ the radius of the circle;

$t =$ the time of moving over a side of the polygon;

$v =$ the velocity with which the particle moves;

$m =$ the mass of the particle;

$Q =$ the value of the impulse; and

$v_1 =$ the velocity imparted by the impulse.

The diagonal BF bisects the angle ABC and passes through the centre of the circle. Prolong it to G, and draw the chords GC and AC. The similar right angled triangles BCG and BEC give

$$\frac{BE}{BC} = \frac{BC}{BG};$$

or,

$$\frac{\frac{1}{2}s}{l} = \frac{l}{2r};$$

$$\therefore s = \frac{l^2}{r}.$$

Multiply both sides of this equation by m and divide by t^2, and it may be written

$$\frac{m\dfrac{s}{t}}{t} = m\frac{\dfrac{l^2}{t^2}}{r};$$

which gives

$$\frac{mv_1}{t} = m\frac{v^2}{r}. \quad . \quad . \quad . \quad (1)$$

According to Article 122, the first member of this equation is the force which, acting during a time t (the time of moving over one side of the polygon), will produce the required velocity v_1; that is, we have

$$F = \frac{mv_1}{t}. \quad . \quad . \quad . \quad (2)$$

But, in order that the motion shall be along the straight line BC, the velocity v_1 must be produced in an instant. Let F' be the force which will produce the velocity v_1 in an element of time $\overline{\Delta t}$; then, according to Articles 122 and 124, we have

$$F' = \frac{mv_1}{\Delta t} = \frac{Q}{\Delta t}. \quad . \quad . \quad . \quad (3)$$

If this force acts upon the particle at all the angles of the polygon, the particle will describe the successive sides of the polygon with a uniform velocity.

The second member of equation (1) is independent of the number of sides of the polygon, and, hence, for the same circle and a constant velocity, the ratio of v_1 to t in the first member will remain constant. If the number of

sides of the polygon be increased, the velocity remaining constant, the time of describing a side will be diminished, hence, v_1, equations (2) and (3), will decrease, and F', in equation (3), will also decrease. Let the number of sides of the polygon be increased until the element of time $\overline{\Delta t}$, during which F' acts, is the same as that occupied by the particle in describing one of the sides of the polygon, and let $\overline{\Delta v_1}$ be the velocity which it will produce in that time; then will the value of F' be

$$F' = m \frac{\overline{\Delta v_1}}{\overline{\Delta t}};$$

and since $\overline{\Delta t}$, $\overline{\Delta v_1}$, and v are simultaneous quantities, equation (1) becomes

$$F' = m \frac{\overline{\Delta v_1}}{\overline{\Delta t}} = m \frac{v^2}{r}. \quad . \quad . \quad . \quad (4)$$

Under these conditions the particle describes the sides of a polygon of an indefinitely large number of sides with a uniform velocity. But the limit of the polygons is the circle; the limit of the fraction $\frac{\overline{\Delta v_1}}{\overline{\Delta t}}$ is, according to equation (4), the constant quantity $\frac{v^2}{r}$; and the limit of F' is some constant, whose value equals the limiting value of the second member of equation (4). But *what is constantly true of a value as it approaches a limit indefinitely, is true of the limit;* hence, calling f the limiting value of F', we have, for the value of f, *when the motion is in the arc of a circle,*

$$f = m \frac{v^2}{r}; \quad . \quad . \quad . \quad . \quad (5)$$

which is the *constant* force which acts towards the centre

ɔllows from this equation that the deflecting *force* *ɛs directly as the mass of the body and the square of* *elocity, and inversely as the radius of the circle.*

4. *To find the value of the central force in terms of* *1ngular Velocity.*

ɩt $\omega =$ the angular velocity, then

$$v = r\omega;$$

equation (5) becomes

$$f = mr\omega^2.$$

5. *To find the value of the Centripetal Force in terms* *e time of a complete revolution.*

ɩe time of a complete revolution is called the Periodic of the motion.

t $T =$ the periodic time; then

$$vT = 2\pi r;$$

$$\therefore v = \frac{2\pi r}{T};$$

h, substituted in equation (5) of Article 313, gives

$$f = m\frac{4\pi^2 r}{T^2}.$$

3. Centrifugal Force of Bodies of finite size.— n_1, m_2, etc., be the masses of the particles of the body, , etc., be their respective distances from the axis of ution, M the total mass of the body, and $\bar{r}$ the point ɩich, if the whole mass of the body were concentrated, ɔentrifugal force would remain the same; then we

$$M\bar{r}\omega = (m_1r_1 + m_2r_2 + \text{etc.})\omega = \omega\Sigma mr;$$

$$\therefore \bar{r} = \frac{\Sigma mr}{M};$$

hence, according to Article 216, *the centrifugal force will be the same as if the whole mass of the body were concentrated at its centre of gravity and revolved with the same angular velocity.*

Problems.

317. *Find the* **centrifugal force on the equator** *due to the revolution of the earth on its axis.*

The time of the revolution of the earth on its axis is $T = 86{,}164$ seconds of mean solar time;* the equatorial radius of the earth is 20,923,161 feet;† and $\pi = 3.14159$. These values in the equation of Article 315 give

$$f = 0.1112m.$$

The force of gravity exceeds this value; for it is found that a body on the equator weighs (see page 32),

$$W = mg = 32.0902m;$$

hence, if the earth did not rotate, and other things remained as at present, a body would weigh

$$W_1 = W + f = 32.2014m.$$

We also have

$$\frac{f}{W_1} = \frac{0.1112m}{32.2014m} = \frac{1}{289} \text{ nearly};$$

hence, the rotation of the earth causes a diminution of the weight of bodies; the diminution on the equator being $\frac{1}{289}$ of their original weight.

318. *In what time must the earth revolve so that bodies on the equator will weigh nothing?*

---

* The earth makes a complete revolution in less than 24 hours mean time.

† See foot note on page 33.

Let T_1 be the required time, then will f, in Article 315, equal W_1, and we have

$$W_1 = m\,\frac{4\pi^2 r}{T_1^2};$$

and dividing the equation of Article 315 by this equation, calling T the periodic time of the revolution of the earth, we have

$$\frac{f}{W_1} = \frac{T_1^2}{T^2} = \frac{1}{289}$$

$$\therefore\ T_1 = \sqrt{\frac{1}{289}}\,T = \frac{1}{17}\,T = 1\,h.\ 24\,m.\ 42\tfrac{6}{17}\ \text{sec.}$$

319. ***To find the central (centripetal) force necessary to keep* the moon in its orbit.**

The mean time of the periodic motion of the moon is $T = 2{,}360{,}585$ seconds; and the mean distance of the moon from the centre of the earth is $60.361\,R$, in which R is the mean radius of the earth. Calling $R = 20{,}897{,}500$ feet and substituting these quantities in the equation of Article 315, gives, for a unit of mass,

$$f = 0.0089.$$

320. *What is the force of gravity at the moon?*

The force of gravity varies inversely as the square of the distance from the centre of the earth. (See Article 65.) Calling the mean value of g at the surface of the earth 32.246, we have

$$g' = \frac{32.246}{(60.3612)^2} = 0.0088.$$

If the data in the two preceding problems were correct in every particular, the results should be exactly alike, that is, $f = g'$. In this way Sir Isaac Newton tested the law of UNIVERSAL GRAVITATION.

321. A locomotive *whose weight is W tons runs around a horizontal curve whose radius is r feet, with a velocity of V miles per hour; required the centrifugal force; that is, the pressure against the outer rail.*

These quantities, reduced to the units of equation (5), Article 313,—that is, to feet, pounds, and seconds—and substituted in that equation, give

$$f = \frac{2{,}000\ W}{32\frac{1}{6}} \cdot \frac{\left(\frac{5280}{60 \times 60} V\right)^2}{r}$$

$$= 130\ \frac{W (\textit{in tons}) \times V^2 (\textit{in m. pr. h.})}{r\ (\textit{in feet})}\ \textit{lbs., nearly.}$$

322. To find the elevation of the outer rail *so that the resultant pressure of a locomotive, as it passes around a curve, will be perpendicular to the plane of the track.*

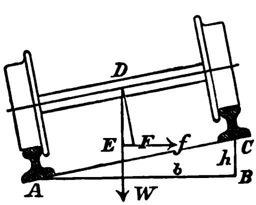


FIG. 145.

Let DE represent the weight, W, of the locomotive, and EF the centrifugal force; then will DF be the resultant of these forces. The elevation of the outer rail must be such that DF will be perpendicular to the line joining the tops of the rails, which, in practice, will be parallel to the line AC. Draw AB horizontal and CB vertical.

Let $b = AB$; $h = BC$; $r =$ the radius of the curve; and $v =$ the velocity of the locomotive.

Then the similar triangles DEF and ABC give

$$DE : EF :: AB : BC;$$

or,

$$W : f :: b : h;$$

$$\therefore h = \frac{f}{W} b.$$

Substitute the value of f, equation (5), Article 313, and $W = mg$, and we have

$$h = \frac{v^2}{g} \cdot \frac{b}{r},$$

in which v and g are both measured in feet per second, and b and r in feet. In practice, the elevation is usually so small that b is taken equal to AC, the gauge of the track. It will be seen that the elevation is independent of the weight of the body, and that it varies as the square of the velocity.

323. To determine the motion of a Conical Pendulum.—*A conical pendulum* is a heavy body revolving about a vertical axis. The governor of a steam engine is an example. The forces which act upon the body B are the force of gravity, which equals the weight of the body, the tension of the piece BA, and the force which produces the rotation. Let the body be at rest and a force, equal to the centrifugal force, act upon the body; then will the relation of the parts be the same as before.

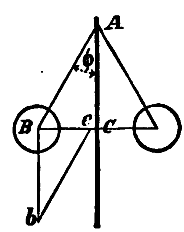


FIG. 146.

Let $W = Bb$, and resolve it into two forces, one, Bc, horizontal; the other, bc, parallel to BA; then will the former represent the centrifugal force, and the latter the tension on BA. Let m = the mass of the body and v its velocity, then, according to Article 313,

$$Bc = m\frac{v^2}{BC} = m\frac{v^2}{AB \sin\phi}.$$

We also have

$$Bc = W \tan Bbc = W \tan \phi;$$

$$\therefore m \frac{v^2}{AB \sin \phi} = mg \tan \phi;$$

from which we find

$$v = \sqrt{AB \, . \, g \sin \phi \tan \phi}.$$

Let $T =$ the time of one revolution, then

$$T = \frac{2\pi \, . \, BC}{v}$$

$$= \frac{2\pi . AB . \sin \phi}{\sqrt{AB \, . \, g \sin \phi \tan \phi}}$$

$$= 2\pi \sqrt{\frac{AB}{g} \cos \phi};$$

from which we find

$$\cos \phi = \frac{gT^2}{4\pi^2 . AB};$$

hence, the angle is independent of the mass of the body.

In the governor the balls are not permitted to move out *freely*, but are required to overcome a resistance. The resistance may be reduced to an equivalent horizontal force applied at B. Let F be this force; then will

$$f = W \tan \phi + F;$$

$$\therefore F = m \frac{v^2}{BC} - W \tan \phi$$

$$= W \left(\frac{v^2}{g \, . \, AB . \sin \phi} - \tan \phi \right).$$

EXAMPLES.

A body revolves about a point in a horizontal plane to which it is attached by means of a cord; required the velocity of the body so that the tension on the cord shall be twice the weight of the body.

In the preceding example, if the radius of the circle is 2 yards, what must be the number of revolutions per minute so that the tension on the string will be three times the weight of the body?

A body is attached to a point by means of a cord 2 feet long, and revolves uniformly in a vertical circle; required the number of revolutions per minute so that the tension of the cord shall be zero when the body is at the highest point of the circle.

In the preceding example, what will be the tension of the cord when the body is at the lowest point of the circle?

A body is placed on the inside of the rim of a wheel which revolves about a vertical axis; if the weight of the body is W, the radius of the wheel R, and the coefficient of friction μ, what must be the number of revolutions of the wheel per minute so that the friction due to the centrifugal force will just prevent the body from sliding vertically downward?

A body is placed in a groove in a horizontal disc, and attached to the centre of the disc by means of a string 30 inches long. The disc is revolved about a vertical axis through its centre at the rate of 250 revolutions per minute; the coefficient of friction between the body and disc being 0.15, what will be the tension of the string, the groove being radial.

7. A car runs around a curve whose radius is 2,500 feet; if the rails are in a horizontal plane, and a body rests on the floor of the car between which and the body the coefficient of friction is 0.10, what must be the velocity of the car in miles per hour so that the body will be just on the point of sliding outward?

8. What must be the elevation of the outer rail so that a car, moving on a curve whose radius is 3,000 feet, with a velocity of 30 miles per hour, shall press equally upon both rails, the distance between the rails being 4 feet 8 inches.

9. A weight is suspended from a point in the roof of a car by means of a string 6 feet long; the car runs around a curve, whose radius is 4,000 feet, at the rate of 40 miles per hour; how much will the string deviate from a vertical on account of the deflecting force?

10. In Fig. 146, if AB is 15 inches, and the body makes 100 revolutions per minute, find the angle BAC, and the distances AC and BC.

11. In Fig. 146, if the resistance which the centrifugal force has to overcome is 4 lbs., acting horizontally when reduced to the point B, the weight of B 5 lbs., and the length AB, 14 inches; what must be the number of revolutions per minute so that the resistance will be overcome when the angle BAC is 10 degrees?

12. If a grindstone whose diameter is 4 feet, thickness 4 inches, tenacity 600 pounds per square inch, revolves about an axis through its centre, how many revolutions must it make per minute to produce rupture along a diameter, no allowance being made for the eye.

EXERCISES.

If a body moves in a curved line which is not the arc of a circle under the action of a central force, will the deviating force be the same as the central force?

Why will the water in a rotating vessel be highest around the outside of the vessel? Will this be true of anything besides water; as grain, or pebbles?

If the rotation of the earth were to cease, about how much would the water in the ocean be raised at the poles, and how much would it be depressed at the equator, the earth being considered as fluid?

Does the centrifugal force have any effect upon bodies or particles below the surface of the earth, as in a deep mine, for instance?

If the earth were a hollow sphere and water were *thrown* into the hollow, where would it come to rest?

If the earth were to revolve on its axis once in 84 minutes, what would happen to bodies on the equator? Would the cohesion of the parts prevent their being thrown off?

If the moon retained a circular orbit, but should revolve around the earth every 15 days, would it be nearer or more remote from the earth than at present?

Why do those planets near the sun go around it in less time thar those more remote?

Does elevating the outer rail *destroy* the centrifugal force of a moving train?

. Water is put on the face of a grindstone and the stone revolved so rapidly that the water flies off; does it go off radially or tangentially?

. Do the centripetal or centrifugal forces have anything to do with the velocity with which the stone leaves the sling?

. Clothes may be partially dried by placing them within a perforated vessel which is made to revolve very rapidly; explain the principle.

. Are bodies at the poles of the earth affected in their weight on account of the rotation of the earth? Show why they weigh more there than they would if the earth ceased to rotate?

. If a train of cars runs around a circular track in which both rails are in the same horizontal plane, is there any danger of the cars being overturned by the centrifugal force?

CHAPTER XVII.

SOLUTION OF PROBLEMS IN WHICH THE INTENSITY OF THE FORCE VARIES DIRECTLY AS THE DISTANCE FROM THE CENTRE OF THE FORCE, AND IS ATTRACTIVE.

324. When a perfectly elastic solid is pulled or compressed, or distorted in any manner, the force which resists distortion varies directly as the amount of distortion, if the elastic limits are not exceeded. See Article 131. This *law*, in which the force varies directly as the distance, holds good in several other problems.

General Formulas.

325. To find the Velocity.—*If a body starts from rest and is acted upon by a force which varies directly as the distance from the centre of the force; required the velocity of the body when it reaches the centre of the force, and also the velocity at any point of the path.*

Let A, Fig. 147, be the origin of the force, B the point where the particle starts, and the line AB the path along which the particle moves. Erect BC to represent the intensity of the force at B, and draw the straight line AC; then will any ordinate bc represent the force acting upon the particle when it is at b. The body, starting from rest at B, will be constantly accelerated until it arrives at A,

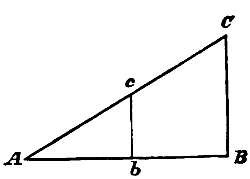


FIG. 147.

but the rate of acceleration will continually decrease from B to A, because the force decreases in intensity.

To represent the velocity approximately by a geometrical construction, divide the space AB, Fig. 148, into equal small spaces, Bf, fg, gh, etc., and erect ordinates, fa, gc, he, etc. Consider the force as constant while the particle is passing from B to f, and represented by the arithmetical mean between BC and fa. This force will produce a certain velocity, which represent by Bm. Draw mn parallel to AB, and at n, where it intersects af prolonged,

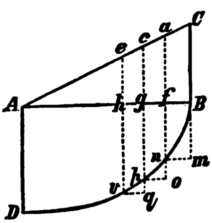


FIG. 148.

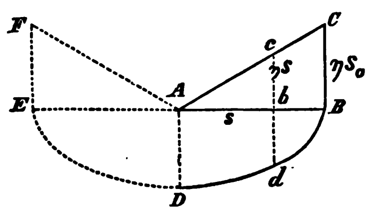


FIG. 149.

draw no to represent the velocity produced by $\frac{1}{2}(af + cg)$, and draw op parallel to AB, to meet cg prolonged at p, and so on. The points B, n, p, etc., will be the vertices of a polygon; and by reducing the spaces Bf, fg, etc., indefinitely, the polygon will approach a curve, BD, as a limit. Any ordinate to this curve, as bd, Fig. 149, will *represent* the velocity of the particle when it has reached the point b of its path.

To find a formula for the velocity, let

η = the intensity of the force at a unit's distance from the centre of the force;

$s_0 = AB$ = the distance of the particle from A when motion begins;

$s = Ab =$ any distance from A;
$v =$ the velocity of the particle at b;
$v_1 =$ the velocity of the particle at A; and
$m =$ the mass of the particle;

then will

$\eta s_0 = BC =$ the intensity of the force at B; and
$\eta s = bc =$ the intensity of the force at b.

The work done by the force upon the particle while moving it from B to A, will be represented by the triangle ABC, see Article 96, and hence, will be

$$\tfrac{1}{2}s_0 . \eta s_0 = \tfrac{1}{2}\eta s_0^2;$$

which will impart to the body an amount of kinetic energy expressed by

$$\tfrac{1}{2}mv_1^2;$$

$$\therefore \tfrac{1}{2}mv_1^2 = \tfrac{1}{2}\eta s_0^2;$$

and

$$v_1 = \sqrt{\frac{\eta}{m}}\, s_0; \quad . \quad . \quad . \quad (1)$$

hence, *the velocity at the centre of the force will vary directly as the distance over which the body moves, and directly as the square root of the intensity of the force at a unit's distance from the centre of the force.*

The work done while moving from b to A will be

$$\tfrac{1}{2}\eta s^2;$$

hence, the work done in passing from B to b will be the difference of these, or,

$$\tfrac{1}{2}\eta(s_0^2 - s^2) = \tfrac{1}{2}mv^2;$$

$$\therefore v = \sqrt{\frac{\eta}{m}}\sqrt{s_0^2 - s^2}, \quad . \quad . \quad (2)$$

which gives the velocity at any point of the path.

The last equation may be written

$$mv^2 + \eta s^2 = \eta s_0^2; \quad . \quad . \quad . \quad (3)$$

ıd since v and s are variables, and all the other quanti-es constants, it is the equation of an *ellipse*. Hence, the ırve BDE, Fig. 149, is an ellipse, of which the semi-‹is AB is s_0, and that of AD is found by making $s = o$, ıd finding the value of v. This value of v becomes v_1, ıd is

$$AD = v_1 = \sqrt{\frac{\eta}{m}}\, s_0;$$

ı given in equation (1).

After the particle arrives at A it will, by virtue of the netic energy of the body, pass that point and move on ıtil the force at A overcomes the energy, when it will op and return. The distance AE will equal AB. The ıtire distance BE is called the *amplitude*. This motion called *oscillatory* or *vibratory*.

326. To find the time *of the movement of a particle rom any distance to the centre of the force, when the orce varies directly as the distance from the centre.*

Equation (2) of the preceding article may be written

$$s_0^2 - s^2 = \frac{m}{\eta} v^2;$$

hich may be represented by a right-angled triangle, in hich s_0 is the hypothenuse, s one side, and $\sqrt{\frac{m}{\eta}}\, v$, the other ide. If with A as a centre, and a radius $AD = s_0$, a uadrant be described, then will BD, the sine of the arc $'D$ to the radius of the arc, and AB, the cosine of the

same arc, *constantly* represent the relation between the space s, and the $\sqrt{\frac{m}{\eta}}$ times the velocity v.

The velocity is constantly varying; hence, at any instant, we have, according to Article 10,

$$\overline{\Delta t} = \frac{\overline{\Delta s}}{v}, \quad . \quad . \quad . \quad (1)$$

in which $\overline{\Delta s}$ is the increment of space passed over in the corresponding increment of time. In Fig. 151, let $AD = s_0$, $BD = s$, and $AB = \sqrt{\frac{m}{\eta}}\, v$.

On the line DB, take Db to represent $\overline{\Delta s}$, and draw ba parallel to AB to meet the tangent Da drawn through D.

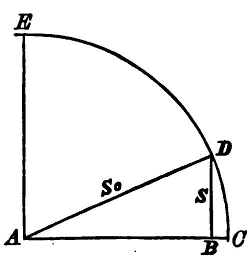


FIG. 150.

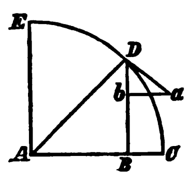


FIG. 151.

The triangles Dba and ABD are similar, having the sides of the one perpendicular respectively to the sides of the other. Hence, we have

$$AB : AD :: Db : Da;$$

or

$$\sqrt{\frac{m}{\eta}}\, v : s_0 :: \overline{\Delta s} : Da;$$

$$\therefore Da = \sqrt{\frac{\eta}{m}}\, s_0 \frac{\overline{\varDelta s}}{v}; \quad . \quad . \quad (2)$$

combined with equation (1), gives

$$Da = \sqrt{\frac{\eta}{m}}\, s_0 . \overline{\varDelta t};$$

$$\overline{\varDelta t} = \frac{Da}{\sqrt{\frac{\eta}{m}}\, s_0}. \quad . \quad . \quad . \quad (3)$$

)a be diminished indefinitely, it will approach the a limit; hence, an element of the time equals an ıt of the arc divided by $\sqrt{\frac{\eta}{m}}\, s_0$; and the time of ʒ from C to A will equal the quadrant CDE, d by $\sqrt{\frac{\eta}{m}}\, s_0$. Let t_1 be the required time, then

$$t_1 = \frac{\frac{1}{2}\pi s_0}{\sqrt{\frac{\eta}{m}}\, s_0} = \frac{1}{2}\,\pi\,\frac{\sqrt{m}}{\sqrt{\eta}} \quad . \quad . \quad (4)$$

, this remarkable result, that *the time of the move- f a particle from any point to the centre of the force, the force varies directly as the distance from the is* INDEPENDENT OF THE DISTANCE. The times, there- re *isochronous.*

ıce, also, *the time will be the same for all distances the centre;* and for the same body *it will vary in-* ı as the *square root of the intensity of the force at 's distance from the centre of the force.*

$t =$ the time of moving through the amplitude;

$$t = 2t_1 = \pi\,\frac{\sqrt{m}}{\sqrt{\eta}}. \quad . \quad . \quad . \quad (5)$$

Problems.

327. Simple Pendulum.—*A simple pendulum* is a material particle, suspended by a mathematical line, and swinging under the action of gravity.

Let O be the point of suspension, P the position of the particle, and $w = mg$ the weight of the particle. Resolve the weight w into two components, one, ba, parallel to OP; the other, Pb, perpendicular to OP, which will also be tangent to the arc BA. The line OA being vertical, the angle baP will equal that at O. The component ab will be resisted by the tension of the cord OP, and the motion will be produced by the component bP. We have

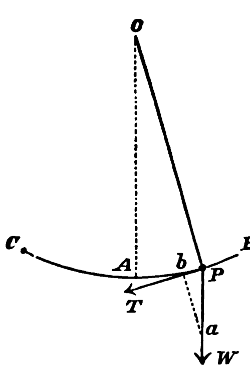


FIG. 152.

$$\begin{aligned} bP &= w \sin a, \\ &= mg \sin O, \\ &= mg\frac{AP}{OP}, \textit{ nearly}, \end{aligned}$$

when the angle O is small. As OP is a constant radius, it follows that, for small angles of oscillations, the moving force varies directly as the distance AP, of the particle from the lowest point A. Hence, the time of an oscillation may be determined from equation (5). For this purpose we must find the value of η. We have

$$\eta = \frac{bP}{AP} = \frac{mg\frac{AP}{OP}}{AP} = \frac{mg}{OP};$$

which, substituted in equation (5), and making $OP = l$, gives

$$t = \pi \sqrt{\frac{l}{g}},$$

s the value given in Article 67. In practice a nd pendulum is always used.

Compound Pendulum.—Any body of finite size, g under the action of gravity, is a compound pen- It is shown, both by analysis and by experiment, re are always two points in a compound pen- ıbout which the body will oscillate in the ne, and the distance between these points is ;th of an equivalent simple pendulum. the body be suspended at A, and the num- vibrations be noted, then there is another ?, at which, if it be suspended, it will vibrate ıme time, and the distance AB will be the of the equivalent simple pendulum. The is called the centre of suspension and B the cen- cillation. It requires *very accurate* measurements, y close observations to determine the length of the m which oscillates in a certain time, but it has been ıany times in different places and the results have :orded. Having found this result, the length of a m which will oscillate once in a second may be s follows:

FIG. 153.

= the length of the pendulum which will oscillate a second; then, according to the equation of the ıg article, we have

$$1\ sec. = \pi \sqrt{\frac{l_s}{g}}.$$

ing this equation by the former one, and solving ives

$$l_s = \frac{l}{t^2}.$$

From the preceding equation we have

$$g = \pi^2 l_s;$$

by means of which the acceleration due to gravity may be found.

329. Length of the Second's Pendulum.

TABLE GIVING THE LENGTH OF THE SECOND'S PENDULUM AT DIFFERENT PLACES ON THE EARTH, AND THE ACCELERATION DUE TO GRAVITY AT THOSE PLACES.

Observer.	Place.	Latitude.	Length of seconds pendulum in inches.	Accelerating force of gravity; feet per second.
Sabine..........	Spitzbergen......	N. 79°50′	39·21469	32·2528
Sabine..........	Hammerfest	70°40′	39·19475	32·2363
Svanberg........	Stockholm	59°21′	39·16541	32·2122
Bessel..........	Königsberg......	54°42′	39·15072	32·2002
Sabine	Greenwich	51°29′	39·13983	32·1912
Borda, Biot, and Sabine........	Paris	48°50′	39·12851	32·1819
Biot	Bordeaux	44°50′	39·11296	32·1691
Sabine..........	New York........	40°43′	39·10120	32·1594
Freycinet	Sandwich Islands	20°52′	39·04690	32·1148
Sabine..........	Trinidad........	10°39′	39·01888	32·0913
Freycinet	Rawak..........	S. 0° 2′	39·01433	32·0880
Sabine and Duperrey........	Ascension.......	7°55′	39·02363	32·0956
Freycinet and Duperrey	Isle of France...	20°10′	39·04684	32·1151
Brisbane and Rumker.......	Paramatta	33°49′	39·07452	32·1375
Freycinet and Duperrey	Isles Malouines..	51°35′	39·13781	32·1895

330. To find the number of seconds lost by a clock *when carried to a given height above the surface of the earth.*

Let the clock on the surface of the earth indicate mean solar time, then in one day it indicates 86,400 seconds; when taken to a height above the earth, the vibrations of the pendulum will be slower, because the force of gravity will be less.

Let $N = 86{,}400$, $N_1 =$ the number of seconds indicated by the clock when at a height h, $t =$ the time of one vibration on the surface, $t_1 =$ the time of one vibration at the height h, and $r =$ the radius of the earth. The length of the pendulum remaining the same, we have

$$t : t_1 :: \frac{1}{\sqrt{g}} : \frac{1}{\sqrt{g_1}} :: \frac{1}{N} : \frac{1}{N_1}.$$

But

$$\sqrt{g} : \sqrt{g_1} :: \frac{1}{r} : \frac{1}{r+h}$$

$$:: N : N_1;$$

hence,

$$N_1 = \frac{rN}{r+h};$$

which, subtracted from N, gives

$$N - N_1 = \frac{h}{r+h} N$$

$$= \frac{N}{1 + \frac{r}{h}}.$$

The quantity $N - N_1$ is called the rate of the clock. The quantities r and h must be of the same denomination.

If the loss in a day be known we may find the height, for we find

$$h = \frac{N - N_1}{N_1} r.$$

331. To find the time in which a body would pass through the earth from surface to surface, *if it could pass freely without resistances, the earth being considered as homogeneous.*

If the earth were a homogeneous sphere, the attractive force would vary directly as the distance from the centre of the earth; see Article 78. Hence, if r be the radius of the earth, we have

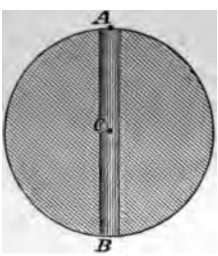


FIG. 154.

$$\eta = \frac{mg}{r};$$

which, substituted in equation (5), Article 326, gives

$$t = \pi \sqrt{\frac{r}{g}}; \quad . \quad . \quad (1)$$

and in equation (1) of Article 325, gives

$$v_1 = \sqrt{gr}. \quad . \quad . \quad . \quad (2)$$

Equation (1), compared with the value of t in Article 327, shows that *the time required for the particle to pass through the earth equals the time of one oscillation, through a small arc, of a simple pendulum whose length equals the radius of the earth.*

Equation (3) of Article 72 gives $v = \sqrt{2gh}$; which, compared with equation (2) above, shows that *the velocity at the centre of the earth will equal $\frac{1}{2}\sqrt{2}$ times the velocity acquired by a body falling freely through a distance equal to the radius of the earth, under the action of a constant*

qual to the force of gravity at the surface of the

Vibration of an Elastic Bar.—Let AB be a ic bar, having a weight P suspended at its lower f the weight be pushed up or down , or be struck, or in any other manner ırbed in a vertical direction, it will oscil- and down ; but, in the case of a solid oscillation will be small. The longi- vibrations of rubber, coiled springs, like, may readily be seen. sverse vibrations, which really follow ıe law, may readily be seen in the case bars, as may be illustrated by a tuning

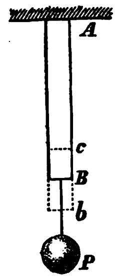


FIG. 155.

ose that AB is the length of the bar when P is led at its lower end, and that by some means it is ed to b. When the disturbing force is removed stic force in the bar will pull the weight up and the l pass the point B and rise to some point as c, in ondition the bar will be in a state of compression. ight will then descend to b, and thus produce a oscillation. Let a force F elongate the bar from and let $Bb = \lambda_1$, then, according to the equation of 131, we have

$$F = \frac{EK}{l}\lambda_1; \quad . \quad . \quad . \quad (1)$$

h E, K, and l are constants, hence, the elongation, λ_1, lirectly as the pulling force, F. The force neces- produce an elongation equal to unity, will be

$$\frac{F}{\lambda_1} = \eta = \frac{EK}{l}; \quad . \quad . \quad . \quad (2)$$

which, substituted in equation (1) of Article 325, making $s_0 = \lambda_1$, gives

$$v_1 = \sqrt{\frac{F\lambda_1}{m}}; \quad . \quad . \quad . \quad (3)$$

or, substituting the value of F, gives

$$v_1 = \lambda_1\sqrt{\frac{EK}{ml}};$$

in which, if $m = P \div g$, we have

$$v_1 = \lambda_1\sqrt{\frac{EKg}{Pl}}. \quad . \quad . \quad . \quad (4)$$

Let λ be the elongation due to P, and we have

$$v_1 = \lambda_1\sqrt{\frac{g}{\lambda}}. \quad . \quad . \quad . \quad (5)$$

The value of η, equation (2), substituted in equation (5) of Article 326, gives

$$t = \pi\sqrt{\frac{ml}{EK}} = \pi\sqrt{\frac{Pl}{gEK}}$$

$$= \pi\sqrt{\frac{\lambda}{g}}; \quad . \quad . \quad . \quad . \quad . \quad (6)$$

hence, the time of an oscillation is independent of the amount of elongation produced by the disturbing force F.

EXAMPLES.

1. What is the length of a pendulum which will vibrate twice in a second?
2. What is the length of a pendulum which will vibrate once in two seconds?
3. A pendulum whose length is 39.1 inches, vibrates

86,420 times in a day; how much must it be lengthened to vibrate once each second?

4. A second's pendulum carried to the top of a mountain lost 45.5 seconds in a day; required the height of the mountain.
5. A pebble is suspended by a fine thread two feet long; required the time of making 5 oscillations.
6. If the radius of the earth be 20,923,161 feet, and g at the surface is 32.0902 feet, what would be the velocity of a body as it passes the centre of the earth if it could pass freely through it?
7. In the preceding example, what would be the time of passing from surface to surface?
8. A prismatic bar, whose cross-section is $\frac{1}{4}$ of a square inch, length 5 feet, coefficient of elasticity 28,000,000 lbs., has two weights suspended at its lower ends, one of 1,000 lbs. and the other of 3,000 lbs. The latter weight suddenly drops off; required the maximum velocity imparted to the remaining weight by the elastic action of the bar.
9. Required the time of one vibration in the preceding example.

[In these problems the mass of the bar is neglected.]

CHAPTER XVIII.

GENERAL PROPERTIES OF FLUIDS.

333. A fluid is a substance in which its particles are free to move among themselves; as air, water, alcohol, etc.

A perfect fluid is a substance in which the particles are *perfectly* free to move among themselves, there being no friction nor cohesion between them, and in which the least force will move any particle in reference to surrounding particles. No such substance is known to exist. Even in the most volatile gas its particles are supposed to offer some resistance between themselves. But the hypothesis of perfect fluidity leads to results which are useful in determining certain formulas applicable to imperfect fluids.

An imperfect or *viscous* fluid is one in which there is a resistance between its particles. There are all grades of viscosity, from that of the most volatile gas to that of solid bodies. Mons. Tresca, a French physicist, proved that even certain solids, as steel and iron, were somewhat viscous. If steel be subjected to an immense pressure by a blunt tool, the metal, in the immediate vicinity of the place pressed, appears to flow like thick tar, or molasses, when either is pressed at a point on the surface. Several armor plates, ten or twelve inches thick, which had been struck by cannon balls, were at the Centennial Exhibition, and were excellent examples of the viscosity of metals.

There are two classes of fluids: *liquids* and *gaseous* bodies; the latter of which includes permanent gases and vapors, and are called *aëriform* bodies.

334. A liquid *is a fluid* in which there is a *slight* cohesion between its particles. Water is taken as a type of liquids. The Greek word for water is ὕδωρ, hence the science of the statical equilibrium of fluids is called *Hydrostatics*, and of their motion, *Hydrodynamics*.

A perfect liquid is a perfect fluid in which there is no cohesion nor repulsion between its particles. The hypothesis of perfect fluidity is assumed unless otherwise stated.

335. Aëriform bodies are those in which the particles mutually repel each other.

If a vessel, made of elastic material, or provided with a piston, be filled with a gas and then enlarged, the gas will constantly fill the vessel. There is no known limit to the expansion of a gas. The space which it occupies depends upon the pressure to which it is subjected.

336. Forces in the Three States of Matter.—The particles which constitute a body act upon each other by forces of attraction and repulsion. Supposing that both forces exist at the same time, the three states of matter—solid, liquid, and gaseous—may be defined by the relations which these forces bear to each other. Thus, a solid is a body in which the force of attraction greatly exceeds that of repulsion; a liquid, one in which the force of attraction equals that of repulsion; and a gaseous body, one in which the repulsion constantly exceeds the attraction.

Many solids may be reduced to liquids by means of heat; the amount of heat depending upon the degree of attraction existing between the particles. Thus, ice, lead, zinc, iron, etc., are examples; and in some cases a substance may be made to assume the three states, of which carbonic acid is a well-known type. The repulsive force may be considered as the effect of heat.

337. Law of Equal Pressures.—*The pressure at any point of a perfect fluid at rest is equal in all directions.*

If it were not equal in every direction the particle at that point would move in the direction of the resultant of the forces.

338. Normal Pressure.—*The pressure of a perfect fluid at rest upon the surface of a vessel which contains it is normal to the surface.*

For, if it were not, it could be resolved into two components, one of which would be tangential to the surface and the other normal to it, and the former would produce motion.

339. Equal Transmission of Pressures.—*If a vessel contain a perfect fluid at rest, and the fluid be destitute of weight, the pressure will be the same at all points within the vessel.*

For the pressure against the fluid arises from the reaction of the sides of the vessel which contains the fluid, and if there was a greater pressure at any point above than at any other point, motion would result. Gases are so light, that, for small quantities under pressure, their weight may generally be neglected. Liquids would be without weight if subjected to the conditions given in Article 77.

340. *The pressure upon the lower part of a vessel which contains a heavy fluid is greater than that at the upper part.*

A heavy fluid is one that has weight. The force of gravity acts downward on each particle, producing a downward pressure. This will be equally transmitted to every point below it, but not above it; hence, the pressure below any particular particle will exceed the pressure above it. The downward pressure follows the same general law as

that of a pile of blocks; the pressure increases from the top to the bottom.

341. *In a heavy, perfect fluid, every pressure, except that due to the weight, will be transmitted to every part of the vessel without diminution of intensity.*

This may be shown experimentally by means of a closed vessel provided with pistons, as in Fig. 156. Let the vessel be filled with a fluid and the pressure upon the pistons be noted. Then, if any piston be pressed inward it will be found that, to prevent the other pistons from moving outward, the same pressure per square inch must be applied to them as to the first piston. In other words, the pressure per square inch will be the same on all the pistons after deducting that due to the weight of the fluid.

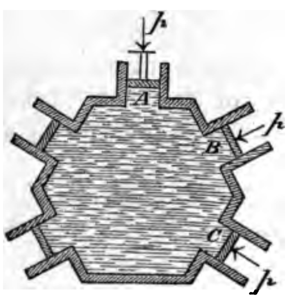


FIG. 156.

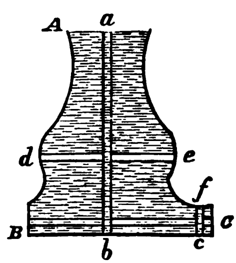


FIG. 157.

342. Vertical Pressures.—*The difference between the pressures on the top and bottom of a vertical prismatic vessel, filled with a heavy, perfect fluid, equals the weight of the fluid.*

Let the narrow strip *ab*, Fig. 157, represent a vertical prism. If there is a downward pressure on the top at *a*, it will, by the principles of equal transmission, be transmitted without diminution of intensity to *b*. The weight

of the particles will also press downward, and since the fluid is supposed to be perfect, the entire weight will be supported by the base b. Hence, the pressure at b will equal the downward pressure at a plus the weight of the fluid.

As a result of this proposition, it follows that, if there is no pressure at a, the pressure at b will equal the weight of the fluid in the prism. If the vessel is filled with a gas it must be closed, and there will necessarily be a pressure at a, but in the case of a liquid there is not necessarily any pressure at that point.

343. In Fig. 157, the vertical pressure upon the base at c is of the same intensity as that at b; for the pressure at b will be transmitted horizontally to c, but at c it will act vertically. The pressure at f will be less than that at c by the weight of the fluid in the prism fc; hence, the pressure on a portion of the surface at f will equal the weight of a prism of the fluid having for its base the area at f, and for its height the distance of the point f below the top of the fluid, *plus* the downward pressure upon the same area at the top of the vessel.

The pressure upon the base of a vessel containing a heavy, perfect fluid is independent of the form of the vessel, and equals the weight of a prism of the fluid having for its base the base of the vessel, and for its altitude the altitude of the vessel, plus *a pressure upon each unit of the base equal to the pressure per unit upon the upper base.*

Let S = the area of the base of the vessel;
a = the altitude of the vessel;
δ = the weight per unit of volume of the fluid;
p = the pressure per unit on the top surface of the fluid; and

$P =$ the total pressure upon the base;

en

$$P = \delta a S + p S.$$

344. Resultant pressure *against the inside of a ves- containing a heavy fluid.*

In Fig. 157, if the points d and e are in the same hori-ntal, and directly opposite to each other, the pressure on them will be equal and opposite, and similarly for other pairs of points on the inside of the vessel; hence e *resultant* pressure upon the whole in-rior surface is zero.

If a hole be cut in one side of the vessel, en, as the fluid is being discharged, there ill be no pressure on that part of the ves-l, and the pressure on the part directly posite will tend to move the vessel in the rection of the pressure. If the vessel be spended by a small cord the effect of this essure may be observed.

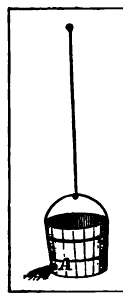


FIG. 158.

It is on this principle that sky-rockets are nt into the air. The pressure due to the arning of the powder acts upward against the rocket, and ownward against the air, and the former pressure raises e rocket.

345. Resolved Pressures.—Let the normal pressure n $ABCD$ be uniform and equal P per unit, and let θ be the in-ination of the surface to the ertical; then will the horizontal omponent of the pressure per nit be

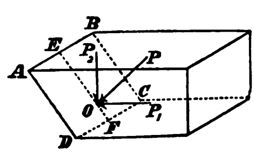


FIG. 159.

$$P_1 = P \cos\theta.$$

Let $S =$ area $ABCD$, and $S_1 =$ the projection of the area on a vertical plane passing through AB; then

$$S_1 = S \cos\theta,$$

and the normal pressure upon the surface S_1 will be

$$P_1 . S_1 = P \cos\theta . S \cos\theta;$$

hence, *the horizontal component of the pressure against an oblique surface equals the horizontal pressure against the vertical projection of the same surface.*

346. Resultant Pressure on a solid body immersed in a heavy fluid.—The resultant of the horizontal pressures will be zero; for the pressure on one side of the body projected in a vertical plane (Article 345) will equal that on the other side projected on the same plane.

If the body be divided into small vertical prisms bc, the vertical component of the pressure at c will exceed that at b by an amount equal to the weight of a prism of the fluid whose volume equals that of the prism bc (Article 342). Hence, *the total upward pressure upon the body equals the weight of a quantity of the liquid of the same volume as that of the solid.* If the weight of the body be less than this pressure, the body will ascend, as is the case with a balloon rising in the air, and light wood rising in water; but if the body be heavier than an equal volume of the fluid it will fall in it, as in the case of bodies falling in the air, or a stone sinking in water.

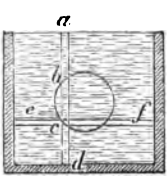


FIG. 160.

347. *The point of application of the resultant pressure* will be at the centre of gravity of the solid, considered as a homogeneous fluid. If the solid be not homogeneous, its centre of gravity will not coincide with the centre of pressure, and if such a body be placed in the fluid so that

ts centre of gravity and the centre of pressure are not in the same vertical, the body will rotate more or less as it rises or falls in the fluid; for the two forces constitute a couple (Article 183), or a couple and a single force (Articles 188, 192). If such a body is not spherical, the combined motions of rotation and translation will generally cause the body to describe a curved path.

348. Equilibrium of Fluids of Different Densities.—*If two fluids which do not mix be placed in two open vessels which communicate with each other, the heights to which they will stand above their common base will be inversely proportional to their densities.*

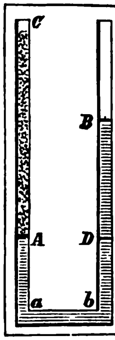


FIG. 161.

Let $CabB$ be the vessel, A the common base, C the surface of one fluid and B that of the other. The portion AD will be in equilibrium, hence the pressure of BD will equal that of CA.

Let $\delta =$ the density of the liquid in BD,
$\delta_1 =$ the density of the liquid in CA,
$h =$ the height BD, and
$h_1 =$ the height CA.

The pressure upon a unit of area of the section at A, due to the fluid in BD, will be (Articles 342, 343)

$$g\delta h,$$

and the pressure due to that in AC will be

$$g\delta_1 h_1;$$

but, there being equilibrium, we have

$$g\delta h = g\delta_1 h_1;$$

$$\therefore \frac{h}{h_1} = \frac{\delta_1}{\delta}.$$

This proposition does not apply to the case in which one of the fluids is lighter than air, for the vessel containing the lighter fluid must be closed at the top.

EXAMPLES.

1. A rectangular closed box, whose depth is 1 ft., breadth 2 ft., and length 3 ft., is filled with a fluid. A cylindrical piston, whose diameter is one inch, is fitted into the top of the box; required the pressure on the bottom and sides of the box which would result from a pressure of 20 lbs. on the piston.
2. If a vessel whose base is 6 square inches and height 8 inches, is filled with water, what will be the pressure upon the base, calling the weight of water $62\frac{1}{2}$ lbs. per cubic foot?
3. The lower part of a vessel is a cylinder whose diameter is 8 inches and height 6 inches; the upper part is also a cylinder whose diameter is 6 inches, and height 4 inches, the vessel is filled with water and subjected to a pressure of 100 lbs. on its upper surface; required the pressure on the base.
4. A cubic foot of wood that weighs 35 lbs. is placed in a vessel of water which weighs 63 lbs. per cubic foot, and the body is prevented from rising by a string fastened to the bottom of the vessel; required the tension of the string.
5. In the preceding example, if the body is free to rise, and there were no resistance to motion from the fluid, what would be its velocity when it has risen 50 feet?

[The acceleration may be found by Article 86 and the velocity by equation (3), Article 24. The result, however, is of no practical value, for the resistance of the liquid will be considerable, varying nearly as the square of the velocity.]

If a cubic block of stone, whose edges are each $1\frac{1}{2}$ ft. and weight 180 lbs. per cubic foot, is suspended in water and held by a cord, what will be the tension of the cord, the water weighing $62\frac{1}{2}$ lbs. per cubic foot?

One half of a prismatic bar is composed of wood and the other half of iron; the iron being 8 times as heavy as the wood, and the whole immersed in a liquid whose weight is 65 lbs. per cubic foot, at what distance from the middle must a cord be attached so that the bar may *rest* in a horizontal position?

If a prismatic tube is bent as in Fig. 161, and filled with mercury to a height DA, how many inches of water must be placed in the tube AC to depress the mercury three inches, the weight of mercury being $13\frac{1}{2}$ times that of water.

EXERCISES.

If a vessel filled with water were placed at rest in a hollow space at the centre of the earth (see Article 77), and the vessel should suddenly vanish, would the liquid disperse? would it remain in the same form as that of the vessel before it vanished?

In the preceding exercise, if the vessel were filled with a gas, what would become of it if the vessel should vanish?

If a pail were filled with tar would the pressure on the bottom of the pail equal the weight of the tar?

How much less will a heavy body weigh in air than it will in a vacuum?

What must be the weight of a body so that it will neither rise nor fall in air?

CHAPTER XIX.

SPECIFIC GRAVITY.

349. Definitions.—The specific gravity of a body is ratio of the weight of the body to the weight of an eq volume of some other body taken as a standard. specific gravity of the standard is taken as unity.

Distilled water is generally taken as the standard comparison for solids and liquids, and atmospheric air aëriform bodies, but both of these substances change t volume for every change of temperature and of press It is necessary to fix a standard temperature and press Some writers have assumed 60° F. for the standard perature for water, while others have taken it at 38.75 assuming that water at that temperature has its n mum density. We will assume the latter as the temp ture for the standard, although the exact temperature responding to the maximum density of water is positively known, it being fixed by some at 38.85° F., by others at 39.101° F. The pressure of the air is d mined by means of a barometer, and, at the level of sea, equals that of a column of mercury about 29.92 in high. When the pressure and temperature are known specific gravity may be reduced to the standard.

The specific gravity of air at 32° F., with the baron at 30 inches, is about $\frac{1}{773}$, water being unity. The relation being established, all substances, including and vapors, may be compared directly with water standard.

[The term *density*, as used in Mechanics, is not identical with that of *specific gravity*, although the ratio of the specific gravities of two bodies is the same as that of their densities. The specific gravity of a cubic foot of distilled water is unity, but its density is $62\frac{1}{4}$ lbs. + $32\frac{1}{6}$, see Article 85.]

350. To find the Specific Gravity of a Body more nse than that of Water.—Weigh the body in a vacuum l then in the standard water; let = the former weight and w_1 the lat- ; then, according to Article 346, the ight of a quantity of water equal in ume to that of the body will be

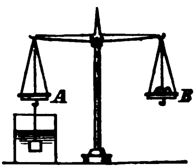


FIG. 162.

$$w - w_1,$$

l, according to the preceding article, the specific gravity l be

$$s = \frac{w}{w - w_1} = \frac{absolute\ weight}{loss\ of\ weight}.$$

351. To find the Specific Gravity of a Body less nse than Water.—Attach it to a body B, which will ıse it to sink in the water, and let

w = the absolute weight of the given body,
w_1 = the absolute weight of the body B,
w_2 = the absolute weight of both bodies,
w_1' = the weight of B in water, and
w_2' = the weight of the combined bodies in water.

en

$w_1 - w_1'$ = loss of weight of B,
$w_2 - w_2'$ = loss of both bodies, and
$(w_2 - w_2') - (w_1 - w_1')$ = loss of weight due to the 'en body, which equals the weight of a mass of water of

the same volume as that of the given body. Substitute in this expression

$$w_2 = w + w_1,$$

and it becomes

$$w + w_1' - w_2'$$

$$\therefore s = \frac{w}{w + w_1' - w_2'}.$$

352. To find the Specific Gravity of a Liquid.—Weigh a body in a vacuum, the water, and in the required liquid.

Let w = the weight of the body in a vacuum,
w_1 = the weight of the same body in water, and
w_2 = the weight of the same body in the liquid;

then

$w - w_1$ = the weight of an equal volume of water, and

$w - w_2$ = the weight of an equal volume of the liquid;

but the volumes being equal, we have, from the definition,

$$s = \frac{w - w_2}{w - w_1}.$$

If an empty bottle, whose weight is w, weighs, when filled with water, w_1, and, when filled with the liquid, w_2, we have

$$s = \frac{w_2 - w}{w_1 - w};$$

which is the same as the preceding formula.

353. To find the absolute Weight of a Body.—Weigh the body in air and in water.

Let w_1 = the weight in air,
w_2 = the weight in water, and
w = the required weight in a vacuum;

then

$w - w_1$ = the weight of a mass of air equal in volume to that of the body, and
$w - w_2$ = the weight of an equal volume of water;

then, if s be the specific gravity of air compared with water as a standard, we have

$$s = \frac{w - w_1}{w - w_2},$$

$$\therefore w = \frac{w_1 - sw_2}{1 - s}.$$

But as s is very small, less than $\frac{1}{700}$, the value of w for most *solids* will be very nearly equal to w_1; hence, for most practical purposes, the weight in air may be used instead of the absolute weight.

354. Specific Gravity of a Soluble Body.—Find its specific gravity in respect to some liquid in which it is not soluble, then find the specific gravity of the liquid in reference to water. Let

s = the specific gravity of the liquid in reference to water;
s_1 = the specific gravity of the substance in reference to the liquid; and
s_2 = the specific gravity of the substance in reference to water;

then

$$s_2 = s_1 . s;$$

for if the body is s_1 times as heavy as the liquid, and the liquid s times as heavy as water, then will the substance be s_1 times s times as heavy as water.

355. Specific Gravity of the Air.—Weigh a large globe which is filled with air. Exhaust the air as completely as possible, the degree of exhaustion being determined by a barometric column, and weigh again. Weigh the same filled with water. Determine the *total* weight of the air originally in the vessel, and divide it by the weight of the water.

This explanation is intended to give only a very crude idea of how it may be determined. In determining the specific gravity accurately there are many details, a description of which is not suited to this work.

Hydrometers, or Areometers.

356. Instruments for determining the specific gravity of fluids, are called Hydrometers, or Areometers. They are of two kinds, one in which the weight is constant and the other in which the volume is constant.

357. Areometer of Constant Weight.—When the same instrument is placed in liquids of different densities it will sink to different depths, because the weight of the volumes displaced must constantly equal the weight of the instrument. In Fig. 163, let CD be a tube of uniform size, B a hollow ball, and A a small vessel containing mercury, so as to make the instrument stand upright in the fluid.

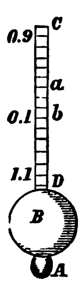


Fig. 163.

To graduate the stem, place sufficient mercury in the vessel A so as to make the instrument float to some definite point and mark it 1.0. Then float the instrument in a liquid whose specific gravity is known to be 1.1 and mark the point to which it sinks 1.1. Divide the intermediate space into 10 equal divisions, and continue the divisions both

above and below as far as desired. This method of equal divisions is not exactly correct, as may be seen from the following formula. Let

V = the volume of the part immersed in water;
v = the volume included between two consecutive divisions of the stem;
D_1 = the density of water;
D = the density of the liquid;
x = the number of divisions between the point marked for water, and the surface when it floats in the liquid, and
s = the specific gravity of the liquid.

Since the weights of the liquids displaced are constant, we have

$$gD_1V = gD(V - vx)$$

$$\therefore s = \frac{D}{D_1} = \frac{V}{V - vx} = \frac{\frac{V}{v}}{\frac{V}{v} - x}.$$

Let the instrument be immersed in a liquid of known specific gravity, say 1.1, and call it s_1, and let $x = 10$ for the space observed, and call its value x_{10}; then we find from the preceding formula

$$\frac{V}{v} = \frac{s_1}{s_1 - 1} x_{10}.$$

Substituting this value in the preceding equation gives

$$s = \frac{\frac{s_1}{s_1 - 1} x_{10}}{\frac{s_1}{s_1 - 1} x_{10} - x};$$

and letting $x = 1, 2, 3$, etc., the values of the specific grav-

ity corresponding to the successive spaces may be puted and marked on the scale; or, if desired, the sp gravities may be assumed and the spaces, x, compute

358. Nicholson's Hydrometer.—This is an areo of constant volume. It consists of a hollow brass cy A, having a small basket, B, at the lowe and carrying a small scale pan, E, at the end. At C is a small vessel of mercu make the instrument float upright. Suf mercury is placed in the vessel so that wit grains in the pan, E, the instrument will to a given notch, D. This instrument m used for determining the specific gravity of or liquids. If the solid is lighter than the vessel at B is inverted so as to force the down into the liquid.

FIG. 164.

To find the specific gravity of a solid, place a quantity of it in the pan E and add weights suffici sink the instrument to D. Then place the substa the basket B, and the additional weights necessary t the instrument to the same mark will be the weight equal volume of water. Let

w = the weight necessary to sink the instr to D in pure water;

w_1 = the weight which must be added to th stance to sink the instrument to the point, when the substance is in the p

w_2 = the weights in the pan E, when the sub is in the basket B, necessary to sink the same point, and

s = the specific gravity of the substance.

Then

$$w - w_1 = \text{the weight of the substance in air,}$$
$$w_2 - w_1 = \text{the loss of weight of the substance in water;}$$

$$\therefore s = \frac{w - w_1}{w_2 - w_1}.$$

To find the specific gravity of a liquid. Sink the instrument to the same depth, D, in water and in the liquid, and let

$W =$ the weight of the instrument,
$w =$ the weight in the pan E when in water, and
$w_1 =$ the weight in the pan when in the liquid;

then

$$s = \frac{W + w_1}{W + w}.$$

Problems.

359. Mechanical Combinations.—*To find the weights of the constituents in a mechanical composition when the specific gravities of the compound and the constituents are known.*

This is a general statement of the noted problem solved by Archimedes, in which he determined the respective amounts of gold and silver in King Hiero's crown.

Let w, w_1, w_2, be the weights of the compound and constituents respectively;
s, s_1, s_2, their respective specific gravities; and
v, v_1, v_2, their respective volumes.

In mechanical combinations we have

$$w = w_1 + w_2; \qquad . \quad . \quad . \quad (1)$$
$$v = v_1 + v_2. \qquad . \quad . \quad . \quad (2)$$

But

$$w = gDv\,;\ w_1 = gD_1v_1\,;\ w_2 = gD_2v_2\,; \quad . \quad (3)$$

which, combined with equation (2), gives

$$\frac{w}{D} = \frac{w_1}{D_1} + \frac{w_2}{D_2}\,; \quad . \quad . \quad . \quad (4)$$

or since their specific gravities are as their densities,

$$\frac{w}{s} = \frac{w_1}{s_1} + \frac{w_2}{s_2}\,; \quad . \quad . \quad . \quad (5)$$

which, combined with equation (1), gives

$$w_1 = \frac{(s_2 - s)s_1}{(s_2 - s_1)s}\,w\,;$$

$$w_2 = \frac{(s_1 - s)s_2}{(s_1 - s_2)s}\,w.$$

360. Chemical Combination.—*Two fluids whose volumes are v and v_1, and specific gravities s and s_1 respectively, on being mixed, contract $\frac{1}{n}$th part of the sum of their volumes; required the specific gravity of the mixture.*

Let $s_2 =$ the specific gravity required, and
$\delta =$ the weight of a unit of volume of water.

If there were no condensation the volume after mixture would be

$$v + v_1\,;$$

but, on account of the condensation, it will be

$$\left(1 - \frac{1}{n}\right)(v + v_1).$$

The sum of their weights before mixture will equal the total weight after mixture, hence,

$$(vs + v_1s_1)\delta = \left(1 - \frac{1}{n}\right)(v + v_1)s_2\delta;$$

$$\therefore s_2 = \frac{n}{n-1} \cdot \frac{vs + v_1s_1}{v + v_1}.$$

361. *In the preceding problem the specific gravity of the mixture being found, required the amount of condensation.*

Solving for $\frac{1}{n}$ gives

$$\frac{1}{n} = 1 - \frac{vs + v_1s_1}{(v + v_1)s_2}.$$

362. To find the Specific Gravity of a Body lighter than Water when weighed in Air.—*A body B_1, whose density is less than water, weighs b_1 grains in air,* and *B_2 in water weighs b_2 grains, and B_1 and B_2 connected, weigh c grains in water. The specific gravity of air being* 0.0013, *required the specific gravity of B_1.*

Let v_1 and v_2 be the volumes respectively of B_1 and B_2, s_1 and s_2 their specific gravities, and δ the weight of a unit of volume of water.

Then

$$(s_2 - 1)v_2\delta = b_2.$$
$$(s_1 - 0.0013)v_1\delta = b_1;$$
$$(s_2 - 1)v_2\delta + (s_1 - 1)v_1\delta = c.$$

From these we find

$$s_1 = \frac{b_1 + 0.0013(b_2 - c)}{b_1 + b_2 - c}.$$

EXAMPLES.

1. A piece of wood weighs 12 lbs., and when annexed to 22 lbs. of lead, whose specific gravity is 11, the whole weighs 8 lbs. in water; required the specific gravity of the wood. *Ans.* $\frac{1}{2}$.
2. Required the specific gravity of a body which weighs 32 grains in a vacuum and 25 grains in water.
3. An areometer sinks to a certain depth in a fluid whose specific gravity is 0.8, and when loaded with 60 grains it sinks to the same depth in water; required the weight of the instrument.
4. A cubic foot of water weighs $62\frac{1}{2}$ lbs.; required the weight of a cubical block of stone whose edges are each 5 ft., its specific gravity being 2.3.
5. If a body sinks $\frac{2}{3}$ of its volume in distilled water, what is its specific gravity?
6. A body, whose weight is 40 grains, weighs 35 grains in water and 32 in an acid; required the specific gravity of the acid.
7. Two pieces of metal weigh respectively 5 and 2 lbs., and their specific gravities are 7 and 9; required the specific gravity of the alloy formed by melting them together, supposing that there is no condensation. *Ans.* 7.474.
8. A compound of gold and silver weighing 10 lbs. has a specific gravity of $s = 14$, that of gold being $s_1 = 19.3$, and of silver $s_2 = 10.5$; required the weight w_1 of the gold and w_2 of the silver.

   Ans. $w_1 = 5.483$ lbs., $w_2 = 4.517$ lbs.
9. If 73 lbs. of sulphuric acid, the specific gravity of which is 1.8485, are mixed with 27 lbs. of water, and the re-

sulting dilute acid has a specific gravity of 1.6321, what will be the amount of condensation?

[Before the formula in Article 361 can be used, it will be necessary to express the quantities in terms of volumes. Let 1 lb. of water be a unit of volume, then will the volume of water be $v = 27$, and of the acid $v_1 = 73 \div 1.8485 = 39.4915$; and in the formula, $s = 1$.]

Ans. 0.0785.

. A body B_2 weighs 10 grains in water, and B_1 14 grains in air, and B_1 and B_2 together weigh 7 grains in water; required the specific gravity of B_1, that of air being 0.0013. *Ans.* 0.8237.

EXERCISES.

If a body floats at a certain depth in a liquid when the vessel which contains it is in the air, will it sink to the same depth when the vessel is in a vacuum?

Why will smoke sometimes rise in the air? Why will it fall at other times?

Will the depth to which a body floats in a liquid be affected by changes in the density of the air?

If a rubber bag containing a gas be made to just sink in a liquid, will a pressure on the surface of the liquid condense the gas? and if so will it have a tendency to rise?

Considering the compressibility of iron and of water, can iron sink so deep in water as to float at that depth? or, in other words, will the water become as dense as the iron?

If water were incompressible, is there any limit to the depth to which a body heavier than water, and also incompressible, will sink in the water? If the body were compressible, is there a limit?

If an egg will sink in pure water, will it float or sink in brine? What must be the condition of the brine that the egg may float between the top and bottom?

Will a vessel of water which contains a fish weigh any more than if the fish were removed?

CHAPTER XX.

HYDROSTATICS.

363. Compressibility of Liquids.—The mechanical properties of liquids are determined on the hypothesis that liquids are incompressible. They are, however, more compressible than most solids. If a cubic inch of water be pressed with fifteen pounds on each and every side, the volume will be diminished $\frac{1}{20000}$, hence one pound to the square inch will diminish the volume $\frac{1}{300000}$. If the water be confined in a perfectly rigid, prismatic vessel, the compression would take place entirely in the direction of the length, and would equal $\frac{1}{300000}$ of the length for every pound per unit of area of the end pressure. Water, therefore, is nearly 100 times as compressible as steel. See Article 130. All other liquids are more or less compressible, yet, for most practical purposes, they may be considered as non-elastic without involving sensible error. Liquids are sometimes defined as *non-elastic fluids.*

The first experiment, to determine the compressibility of water, was made by a philosopher at Florence, Italy. He filled a hollow globe made of gold with water, and then subjected it to a great pressure, thereby flattening it. This diminished the volume, and it was observed that the water oozed out through the pores of the gold; from which he drew the erroneous conclusion that the liquid was not diminished in volume.

364. Free Surface.—The upper surface of a liquid contained in a vessel which receives no pressure, is called

the free surface. The upper surface of water in the atmosphere is pressed downward by the air with about fifteen pounds to the square inch; yet such a surface is often considered as a free surface.

The free surface of small bodies of a perfect liquid at rest may be considered as horizontal; for it will be perpendicular to the direction of action of the force of gravity, Articles 338 and 204; but for large bodies of a liquid it is spherical, partaking of the general form of the surface of the earth.

365. A Level Surface is one which cuts at right angles the resultant of the forces which act upon its particles. Thus, in a vessel filled with a heavy liquid at rest, it is horizontal; in the ocean it may be a surface at any depth and nearly concentric with the free surface; in the second problem below it is a paraboloid of revolution, etc.

Problems.

1. *A vessel is filled with a perfect, homogeneous liquid, and drawn horizontally with a uniform acceleration; required the form of the free surface.*

Let F be the force producing an acceleration f, and M the mass of the liquid. Then, according to Article 86, we have, for the horizontal force,

$$F = Mf = W\frac{f}{g} = cb;$$

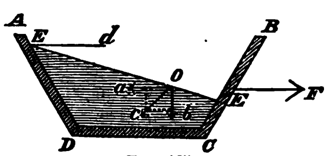


FIG. 165.

and for the vertical force,

$$W = ob.$$

These forces will be uniformly distributed throughout the mass; hence the resultant of the forces on the parti-

cles will be equal and parallel to each other, and also normal to the free surface, Article 338; therefore *the free surface will be a plane.* The level surfaces will also be planes parallel to the free surface.

Let ϕ be the inclination of the free surface to the horizontal, also $= cob$; then

$$\text{tang}\,\phi = \frac{W\frac{f}{g}}{W} = \frac{f}{g}.$$

2. *If a cylindrical vessel containing a perfect, homogeneous liquid be revolved uniformly about a vertical axis, what will be the form of the free surface?*

The vessel may be any solid of revolution, the axis of revolution coinciding with the axis of rotation. Any element of the liquid will be acted upon by two forces; one, the force of gravity acting vertically downward and equal to ob, the weight, w, of the particle; the other, the centripetal force, acting horizontally and radially inward. Let the vessel be at rest and a force equal to the centrifugal force act upon the particles.

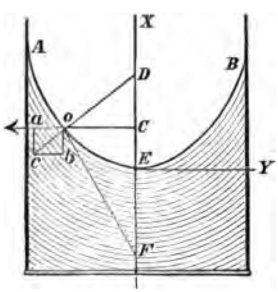


FIG. 166.

Let ω be the angular velocity, and r the distance of any particle from the axis of rotation; then, according to Article 314, the centrifugal force will be

$$oa = mr\omega^2 = \frac{w}{g} r\omega^2 = bc;$$

and the vertical force,

$$ob = w.$$

resultant of these forces must be normal to the free face. Prolong the line of the resultant co, until it ets the axis at D; then, from the similar triangles obc l oCD, we have

$$\frac{DC}{Co} = \frac{ob}{cb},$$

$$\frac{DC}{r} = \frac{w'}{\frac{w}{g} r\omega^2};$$

$$\therefore DC = \frac{g}{\omega^2};$$

t is, the subnormal, DC, is constant. It is shown in the lculus that the parabola is the only curve which posses this property; hence the surface is a paraboloid of olution. All the *level* surfaces are equal paraboloids.

It is also shown in the Calculus that the volume of a aboloid of revolution is one-half that of a circumscribed inder; hence, if the cylinder be at rest, the free sure will be midway between the highest and lowest ints of the paraboloid.

3. *If a perfect, homogeneous mass of liquid be acted on by a force which varies directly as the distance from centre of the mass, what will be the form of the free rface?*

It will be a sphere; for the force at the surface will n be equal and normal at every point of it.

4. *If, in the preceding example, the mass rotates about axis, what will be the form of the free surface?*

The force of gravity will act directly towards the cen- of the mass, and the centrifugal force will act outward,

perpendicular to the axis of rotation, and the resultant of these forces must be normal to the surface. This problem is approximately that of the earth, and its solution involves higher mathematics. The form is, approximately, *an ellipsoid of revolution*, and is often called an *oblate spheroid.*

EXAMPLES.

1. A vessel containing a liquid, whose weight (including the liquid) is 50 lbs., is drawn horizontally by an *effective moving force* (Article 87) of 15 lbs.; required the inclination of the surface to the horizontal.
2. A rectangular box 3 feet long contains a quantity of liquid. If the liquid is one foot deep, what must be the acceleration of the box in a horizontal direction that the free surface at the forward end shall just touch the bottom of the vessel?
3. In the preceding example, if the rear end of the box slopes outward at an angle of 45 degrees, what must be the acceleration of the box so that all the water shall escape by flowing over that end?
4. In Fig. 166, if the vessel is cylindrical and 2 ft. in diameter, and the free surface of the liquid is 3 inches from the top, what must be the number of turns per minute so that the upper edge of the surface shall just reach the edge of the vessel?
5. If the vessel is rotated 30 turns per minute, what will be the equation of the parabola?

Ans. $y^2 = 2\dfrac{g}{\pi^2}x.$

Law of Pressure.

366. The pressure of a perfect, homogeneous liquid *varies directly as the depth below the free surface.*

Since such a liquid is incompressible, we may consider a vertical prism of the liquid as composed of blocks of equal size and weight, placed one above the other. The first block will press with its entire weight upon the second one, and the first and second upon the third, and so on, and since the weights are equal to each other, the pressure upon the succeeding blocks will vary as the number of the blocks, or as 1, 2, 3, etc.

Draw a horizontal line, $1a$, to represent the pressure at 1, then will $2b$, representing the pressure at 2, be twice as long as $1a$; $3c$, three times as long, and so on. In this case

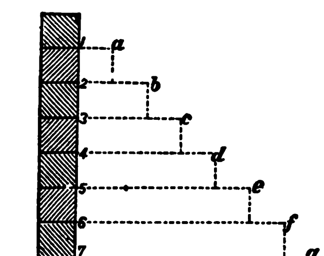


FIG. 167.

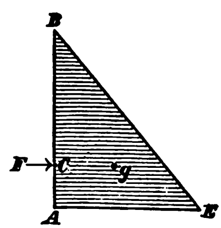


FIG. 168.

the pressure does not increase continuously, but by steps. If now the blocks be divided indefinitely, the steps will become indefinitely small, and ultimately may be represented by Fig. 168, in which AE represents the pressure at A, and any horizontal line drawn from AB to BE, the pressure at that point. Thus far we have considered the vertical pressure only, but in a perfect liquid the pressure

will be the same in all directions, Article 337; hence the truth of the proposition. It follows from this that:

367. *The pressure against an elementary area equals the weight of a prism of the liquid whose base is the area pressed and whose altitude is the vertical distance of the area below the free surface.*

Let $\overline{\varDelta a} =$ be the area,

$h =$ the distance below the free surface, and

$\delta =$ the weight of a unit of volume;

then the pressure will be

$$\delta . \overline{\varDelta a} . h.$$

368. To find the pressure of a liquid against a vertical rectangle *in which one edge coincides with the free surface.*

Let $ABCD$ represent the rectangle, in which the side BC coincides with the surface of the liquid. Draw the horizontal line AE to represent the pressure at A, and draw BE; then will the triangle ABE represent the pressure against the line AB. Complete the triangular wedge $ABCD-FE$; the volume of this wedge will represent the entire pressure against the rectangle.

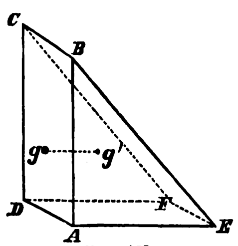


FIG. 169.

Let $\delta =$ the weight of a unit of volume of the liquid;

$h = AB$; $b = AD$.

The pressure on a unit of area at A will be, according to Article 366,

$$AE = \delta h;$$

hence, the volume of the wedge will be

$$bh.\tfrac{1}{2}\delta h = \tfrac{1}{2}\delta bh^2.$$

he liquid be water, this expression becomes

$$31\tfrac{1}{4}bh^2 \; lbs.$$

vhich b and h are in feet.

Ve see from this expression that the *entire* pressure n the free surface downward, varies as the square of distance from the surface.

69. *To find the pressure of a liquid against a vertical tangle when* **the upper edge is parallel to the free face.**

Let $h_1 = BN$; $h_2 = AN$; $b = AD$;

n will the pressure on the rectgle $ABCD$ equal the difference the pressures on $ANMD$ and BNM, or

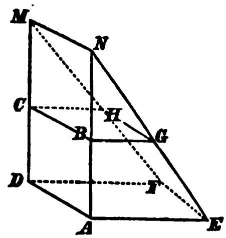


FIG. 170.

$$\tfrac{1}{2}\delta b(h_2^2 - h_1^2).$$

370. Pressure on any Surface.—Conceive the surface, ether plane or curved, to be divided into small areas.

Let s_1, s_2, s_3, etc., be the areas;

h_1, h_2, h_3, etc., their respective distances below the free surface;

$S = s_1 + s_2 + s_3 +$ etc. $= \Sigma s$, be the entire area of the surface pressed by the liquid;

$\bar{x}$, the depth of the mean pressure; and

δ, the weight of a unit of volume.

en, according to Article 367, the pressure upon the sur e will be

$$\delta s_1 h_1 + \delta s_2 h_2 + \delta s_3 h_3 = \delta \Sigma sh;$$

and this also equals the total area into the mean pressure, hence

$$\delta S\bar{x} = \delta \Sigma sh ;$$

$$\therefore \bar{x} = \frac{\Sigma sh}{S} ;$$

hence, according to Article 216, $\bar{x}$ *is the depth of the centre of gravity of the surface.* Therefore, *the total pressure on any surface, S, equals the weight of a prism of the liquid whose base is the area of the surface pressed, and whose altitude is the depth of the centre of gravity of the surface below the free surface.*

Problems.

1. **Triangular Surfaces.**—*To find the pressure against a triangular surface whose base is parallel to the free surface, and whose apex is in that surface.*

Let ABC be the triangle, $b = AC$, $h = BD$, and g the

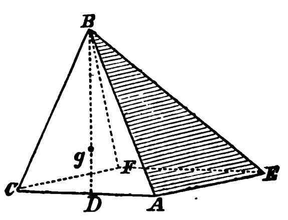


Fig. 171.

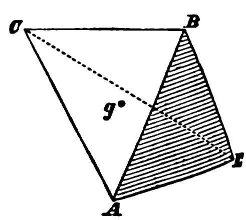


Fig. 172.

centre of gravity of the triangle; then, according to Article 370, we have

$$\delta . \tfrac{1}{2}bh . \tfrac{2}{3}h = \tfrac{1}{3}\delta bh^2.$$

The pressure is also represented by the volume of the pyramid $B-ACFE$, or

$$\delta AE . AC . \tfrac{1}{3}BD = \delta h . b . \tfrac{1}{3}h = \tfrac{1}{3}\delta bh^2,$$

as before.

2. *Let the base coincide with the free surface.* Then we have

$$\delta.\tfrac{1}{2}bh.\tfrac{1}{3}h = \tfrac{1}{6}\delta bh^2.$$

This may also be represented by a triangular pyramid whose base is ABC, Fig. 172, and whose altitude is $AE = \delta h$.

3. **Cones.**—*Find the normal pressure upon the concave surface of a closed cone filled with a liquid ;* (1) *with the axis vertical and apex uppermost :* (2) *axis vertical and cone inverted ;* (3) *with the axis horizontal ;* (4) *pressure on the base in case* (1) *;* (5) *vertical pressure on the concave surface in* (1) *;* (6) *weight of the liquid.*

Let $r =$ the radius of the base, and
$h =$ the altitude of the cone.

Then

(1)	$\tfrac{2}{3}\delta\pi rh\sqrt{r^2+h^2}$.	(4)	$\delta\pi r^2h$.
(2)	$\tfrac{1}{3}\delta\pi rh\sqrt{r^2+h^2}$.	(5)	$\tfrac{2}{3}\delta\pi r^2h$.
(3)	$\delta\pi r^2\sqrt{r^2+h^2}$.	(6)	$\tfrac{1}{3}\delta\pi r^2h$.

Observe that the weight equals (4), the downward pressure, minus (5) the upward pressure.

4. **Spheres.**—*A sphere is submerged in a liquid : find the normal pressure upon the external surface* (1) *when it is just submerged ;* (2) *when submerged to any depth ;* (3) *weight of a quantity of the liquid equal in volume to that of the sphere.*

Let $r =$ the radius of the sphere, and
$h =$ the depth of the centre of the sphere below the free surface.

Then

(1) $4\delta\pi r^3$; (2) $4\delta\pi r^2h$; (3) $\tfrac{4}{3}\delta\pi r^3$.

EXAMPLES.

1. In Fig. 170, if the edge MN of the rectangle coincides with the surface of the liquid, and AN is 3 feet, how far from the surface must the line CB be drawn so that the pressure on the two parts shall be equal? *Ans.* 2.121 *feet.*

2. A rectangle whose sides are 1.4 feet and 2.6 feet respectively, is immersed in water with the former side in the surface, and is inclined at an angle of 56° 35′ to the free surface; required the pressures on the parts into which the rectangle is divided by its diagonal.

3. A cylinder whose base is 2 feet in diameter and altitude 3 feet, is filled with water; required the pressure on the concave surface, the pressure on the base, and the weight of the water.

4. A sphere 10 feet in diameter is filled with water; required the normal pressure on the interior surface, and the weight of the fluid.

5. Find the pressure on a rectangular submerged floodgate, $ABCD$, Fig. 170, whose depth, BN, is 10 ft., height of the gate, AB, 3 ft., and width, BC, 2 ft.

6. In the preceding example, find the pressure if the top of the floodgate is also submerged on the opposite side to a depth of 4 feet.

Centre of Pressure.

371. **The centre of pressure** of any surface immersed is the point of application of the resultant of all

the pressures upon it. It is, therefore, that point in an immersed surface to which, if a force equal and opposite to the resultant of all the pressures upon it be applied, this force will keep the surface in equilibrium.

372. Rectangles.—Let the surface be a rectangle in which one end coincides with the free surface of the liquid; then will the centre of pressure be at two-thirds the depth of the rectangle. The total pressure may be represented by a wedge whose end is the triangle ABE. Hence the centre of pressure will be at the same depth as that of the centre of gravity of the triangle ABE. Let C, on the vertical line BA, be on a horizontal line through the centre of gravity, g, of the triangle ABE. Then, according to Article 222, BC will be $\frac{2}{3}BA$.

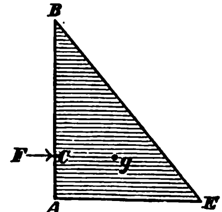


FIG. 173.

373. Submerged Rectangle.—In Fig. 170, the centre of pressure will be at the depth of the centre of gravity of the trapezoid $ABGE$. Since BG and AE are directly proportional to NB and NA, we have, from Example 6, page 146, for the depth required,

$$NA - \tfrac{1}{3}AB\,\frac{NA + 2NB}{NA + NB}.$$

374. Triangles.—The centre of pressure of the triangle in Fig. 171, will be opposite the centre of gravity of the pyramid $B-ACFE$; or

$$Bg = \tfrac{3}{4}BD.$$

375. *The centre of pressure against the triangle ABC, when CB is in the free surface, is at the centre of gravity g, of the wedge $ABC-E$.* To find this point, we observe that this wedge is what remains after removing

the pyramid $BCGF-E$, Fig. 175, from the wedge $BCGF-AE$. The centre of gravity of the large wedge is at one-third its altitude, and of the pyramid at one-

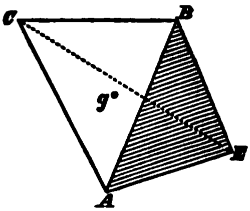


FIG. 174.

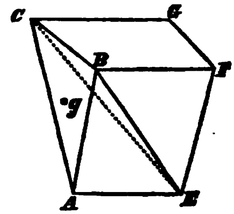


FIG. 175.

fourth its altitude from the base; hence, according to Article 224, we find that the centre of pressure, g, is at *one-half the altitude from the base CB.*

Flotation.

376. Plane of Flotation.—If a body in a liquid is lighter than the liquid, it will float, and the conditions of equilibrium will be determined according to Article 346. The intersection of the plane of the free surface with the floating body is called *the plane of flotation.* The line joining the centre of gravity of the solid and of the displaced liquid is called *the axis of flotation.*

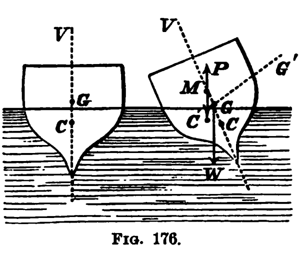


FIG. 176.

377. Conditions of Equilibrium of a Floating Body.—One condition is, according to Article 346, that the weight of the displaced liquid shall equal that of the body. Another is that the axis of flotation shall be vertical.

378. Stable Equilibrium.—In Fig. 176, let C be the centre of gravity of the displaced liquid, and G that of the body when the axis of flotation is vertical; and C' the centre of gravity of the displaced water when the axis is inclined. Let the vertical through C' meet the line GV in the point M. When the body is turned through an indefinitely small angle, the point M is called the *metacentre.* When M is above G the pressure of the fluid upwards along $C'M$, and of the body downwards along the vertical GW, tend to bring the body back to the position in which the axis of flotation will be vertical. Hence the equilibrium is stable when the metacentre is above the centre of gravity of the body. Observing that $C'G$ is a new axis of flotation, it follows that the equilibrium is stable when the axis of flotation turns in a direction opposite to that of the rotation of the body when the position of the body is disturbed.

379. Depth of Flotation.—Let D be the density of the body, V its volume, and s its specific gravity; and D_1, V_1, s_1, the corresponding quantities for the displaced liquid. Then, according to Articles 85 and 349, and the first condition of Article 377, we have

$$gDV = gD_1 V_1;$$

$$\therefore V_1 = \frac{D}{D_1} V = \frac{s}{s_1} V.$$

If the body floats in pure water then $s_1 = 1$ and

$$V_1 = sV.$$

Problems.

1. *Let the body be a right cone with the axis vertical and apex upward; required the depth of flotation.*

Let $r =$ the radius of the base,
$h =$ the altitude, and
$x =$ the depth of flotation.

Then

$$\frac{r}{h}(h-x) = \text{the radius of the plane of flotation.}$$

hence

$$V = \tfrac{1}{3}\pi r^2 h,$$

$$V_1 = \tfrac{1}{3}\pi r^2 h - \tfrac{1}{3}\pi\frac{r^2}{h^2}(h-x)^3;$$

and the equation of the preceding article becomes

$$\tfrac{1}{3}\pi r^2 h - \tfrac{1}{3}\pi\frac{r^2}{h^2}(h-x)^3 = \tfrac{1}{3}\frac{s}{s_1}\pi r^2 h.$$

$$\therefore x = \left(1 - \sqrt[3]{1-\frac{s}{s_1}}\right)h.$$

2. *A rectangular wall whose height is h feet, thickness b feet, and weight of a cubic foot of the masonry δ pounds, resists the pressure of water whose height behind the wall is h_1 feet; will the wall be stable in reference to slipping on its base, or to overturning about its outer edge?*

The pressure of the liquid for a unit of width will be, according to article 368,

$$\tfrac{1}{2} \times 62\tfrac{1}{2}h_1^2;$$

and the centre of pressure, according to Article 372, will be at $\frac{1}{3}h_1$ from the base, hence the *moment* of pressure in reference to the outer edge of the wall will be

$$\tfrac{1}{2} \times 20\tfrac{5}{6}h_1^3.$$

The weight of the wall will be

$$\delta b h;$$

l the moment in reference to the lowest outer point
l be

$$\tfrac{1}{2}\delta b^2 h\,;$$

ıce, the wall will be stable in reference to rotation, if

$$\delta b^2 h > 20\tfrac{5}{6}h_1^3\,;$$

l in reference to slipping on the base, if

$$\mu\delta b h > 31\tfrac{1}{4}h_1^2\,;$$

which μ is the coefficient of friction of the wall on its
ındation; see Article 107.

EXAMPLES.

In Fig. 170, what is the depth of the centre of pressure below *MN*, the end *MN* coinciding with the free surface, and *AN* being 3 feet? *Ans.* 2 *feet.*

In Fig. 170, if *ABCD* is a floodgate, *BN* being 5 feet, and *AB* 10 inches, how far below *B* must a horizontal bar be placed so as to balance the pressure against the gate?

If the height of a rectangular wall be 8 ft., weight of the masonry 180 lbs. per cubic foot, what must be the thickness of the wall to resist overturning from the pressure of water when level with the top of the wall?

A triangular wall whose base is 4 ft., height 8 ft., weighs 120 lbs. per cubic foot; required the height of water which it will sustain in reference to overturning, and leave a coefficient of stability of 2.

[The degree of stability will be determined by assuming 135 lbs. per cubic foot for the weight of the water.]

5. The space between the pistons *H* and *A* is filled with water, and a pressure of 500 lbs. is exerted on the small piston by means of a lever; if the diameter of the small piston is $1\frac{1}{4}$ inches, and of the large one 15 inches, what will be the pressure exerted by the large piston?

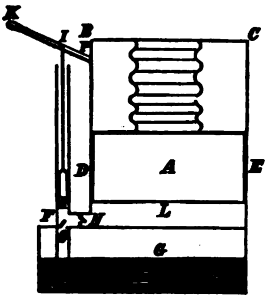


FIG. 177.

[This machine is called a **Hydraulic Press.** It was perfected by one Bramah, who packed the pistons with a leather collar in such a way that the pressure of the water forced the leather against the sides of the cylinder, thus keeping the pistons water-tight.]

EXERCISES.

1. In Figs. 165 and 166, will the surface be the same for mercury as for water, other things being the same?
2. Why do the answers to Examples 1, in Articles 370 and 379, differ?
3. If a vertical square be entirely submerged, will the centre of pressure coincide with the centre of gravity of the square?
4. In the preceding Exercise, if the square be turned about a horizontal line, passing through its centre of gravity, will the depth of the centre of pressure be changed?

CHAPTER XXI.

HYDRODYNAMICS.

380. Mean or Average Velocity.—The velocity is not the same at all points in the cross section of a stream, whether the stream be a river, or canal, or in a pipe or tube. That velocity which, being multiplied by the area of the cross section, will equal the quantity discharged is called the *mean velocity*.

Let $Q =$ the quantity which passes a section,
$S =$ the area of the cross section, and
$v =$ the mean velocity;

then

$$vS = Q;$$

$$\therefore v = \frac{Q}{S}.$$

381. Permanent Flow.—If the same quantity passes all the transverse sections the flow is said to be *permanent*: otherwise it is variable. In a canal the flow would be permanent if there were no wastes from evaporation or leakage; in a pipe without branches the flow will be permanent throughout its length.

382. Variable Velocities.—In a stream of variable sections in which the flow is permanent, the mean velocities are inversely proportional to the transverse sections of the stream.

Let v, S, and Q be the quantities for one section,
v_1, S_1, and Q the quantities for another section;
then, according to Article 380, we have

$$vS = Q = v_1 S_1;$$

$$\therefore \frac{v}{v_1} = \frac{S_1}{S}.$$

383. Problem.—*To find the velocity with which a fect liquid will flow through an orifice in the base vessel.*

We will assume that the particles start from rest point near the orifice, and that the velocity of their is produced by a constant pressure, equal to the weig the water vertically over them (Article 367).

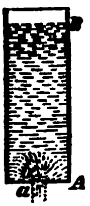


Fig. 178.

Let ab be the short distance through whicl velocity is generated,
$AB = h$, the height of the liquid,
W, the weight of the quantity in the heigl
S, the area of the orifice,
F, the pressure of the liquid above the ori and
v, the velocity of discharge.

Then, since the flow is supposed to be without resist the conditions are essentially the same as Problems 1 ai pages 45 and 46; hence

$$v = \sqrt{\frac{2F.g.ab}{W}}$$

But in this problem we have

$$F = \delta S h,$$
$$W = \delta S.ab;$$

which substituted above gives

$$v = \sqrt{2gh},$$

which is the same as that of a body falling freely thrc a height h; see Article 72.

384. The velocity through an orifice in the side of a vessel will also be

$$v = \sqrt{2gh};$$

in which h is the depth of the orifice below the surface; for the pressure against the side is the same as the vertical pressure at the same depth.

385. Head due to a Velocity.—The preceding equation gives

$$h = \frac{v^2}{2g};$$

in which the height h, corresponding to the velocity v, is called the *head due to the velocity*, or simply *the head.* The corresponding velocity is called *the velocity due to the head.*

386. Vertical Pressure on the top of a Vessel.—If a piston were fitted into the vessel at B, Fig. 178, and a pressure applied to it, the velocity of issue would be increased. To find the resulting velocity, let

$P =$ the pressure on the surface,
$S =$ the area of the upper surface,
$p = P \div S =$ the pressure on a unit of area,
$h_1 =$ a height of liquid which will give a pressure equal to P,
$D =$ the density of the liquid, and
$\delta =$ the weight of a unit of volume of the liquid, $= Dg$ (see the last equation of Art. 85);

then we have

$$P = \delta S h_1;$$

$$\therefore h_1 = \frac{P}{\delta S} = \frac{p}{gD},$$

and the velocity of flow will be

$$v = \sqrt{2g(h + h_1)}.$$

This is called finding an *equivalent head*. If the liquid issues into a fluid more dense than air, there will be a counter-pressure. If h_2 is the head due to the difference of the pressures of the air and fluid, then

$$v = \sqrt{2g(h-h_2)}.$$

387. Pressure of the Air.—When a vessel is in the air it is pressed on the upper surface with nearly 15 pounds to the square inch, which is equivalent to a column of water of the same base and about 34 feet high. If a vessel of any liquid should discharge into a vacuum, this head must be added to the head of the liquid, but in practice the air presses against the issuing stream with the same pressure per unit that it presses against the top, so that the head due to the pressure of the air is not considered.

388. Vertical Jet.—If the issuing jet should be vertically upward, as in Fig. 179, and there were no resistances

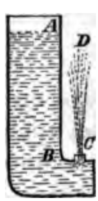


FIG. 179.

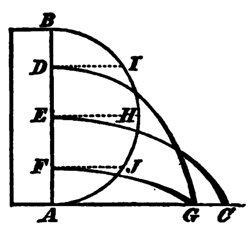


FIG. 180.

from the air or the sides of the orifice, the jet ought to rise as high as the free surface of the liquid in the vessel. But it is found in practice that it always falls short of that height.

389. Orifices in the Side of a Vessel.—If the fluid issues horizontally from an orifice in the side of a vessel, the jet will be subjected to the same law as that of a pro-

jectile thrown horizontally. Let D, Fig. 180, be an orifice in the side of a vessel, and DG the path of the fluid vein. In Article 306 make $x = AG$ and $y = DA$, and we have

$$AG = 2\sqrt{h.DA}.$$

But $h = BD$, hence

$$AG = 2\sqrt{BD.DA}.$$

If on AB, as a diameter, a semicircle AHB be described, and an ordinate DI be erected, then, from geometry, we have

$$DI = \sqrt{BD.DA};$$

hence, the range

$$AG = 2DI;$$

that is, *if a semicircle be constructed on AB as a diameter, the range will be twice the ordinate of the semicircle drawn from the orifice.*

Hence the maximum range AC will result from the flow through an orifice, E, at the middle of the height.

Also, the ranges from two orifices, D and F, equidistant from the centre, E, will equal each other.

390. Oblique Jet.—If the jet be oblique, spouting upward or downward, the range may be determined by the formulas in Article 299, considering the vein as the path of a projectile.

Coefficients of Flow.

391. Coefficients of Contraction.—If the vein issues through a thin plate, the smallest part of the vein will be at a short distance from the orifice. It appears that the particles, as they approach and issue from the orifice, tend

to cross each other's path, and by thus interfering with each other first produce contraction and afterward expansion, as shown in Fig. 181.

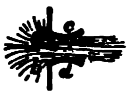


FIG. 181.

Let S be the area of the orifice ab,

S_1, the area of the contracted part cd, and

m_1, the coefficient of contraction;

then

$$S_1 = m_1 S.$$

For a very thin plate, the distance of smallest section of the vein from the orifice will be equal to the radius of the orifice, and the diameter of the smallest section will be 0.8 of the diameter of the orifice, and the coefficient of contraction will be the square of 0.8; hence *for a thin plate* $m_1 = 0.64$.

For an adjutage, that is, for a short tube, $abdc$, whose length is two or three times the diameter of the orifice, attached to the orifice, the fluid vein will just fill it, and the coefficient of contraction will be

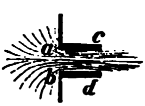


FIG. 182.

$$m_1 = 1.$$

In Fig. 181, if a conically convergent tube form the adjutage, the convergent part being of the form and length of the *vena contracta*, and the smallest diameter of the tube be taken for the orifice, then

$$m_1 = 1.$$

392. Coefficients of Velocity.—In Fig. 179, if h_1 be the height to which the jet will rise, then will the velocity of discharge be

$$v_1 = \sqrt{2gh_1};$$

but the theoretical velocity will be

$$v = \sqrt{2gh};$$

hence

$$v_1 = \sqrt{\frac{h_1}{h}}\, v,$$

from which the value of v_1 may be determined.

Or, from the first equation of Article 389, we have (writing v_1 for v),

$$v_1 = AG\sqrt{\frac{g}{2DA}},$$

by which v_1 may be computed.

Let $m_2 = \frac{v_1}{v} =$ the coefficient of velocity, then

$$v_1 = m_2 v.$$

For a mere orifice in a thin plate, . . . $m_2 = 0.98$
For a short tube, Fig. 182, ... $m_2 = 1.00$

393. Coefficients of Discharge.—The coefficient of discharge is the ratio of the actual discharge to that of the theoretical. Let the quantity which flows through an orifice, or pipe, or stream be measured, and the quantity which should flow be computed; and let

$Q =$ the quantity of theoretical flow,
$Q_1 =$ the quantity of actual flow, and
$m =$ the coefficient of discharge;

then

$$Q_1 = mQ.$$

But the actual flow equals the mean velocity in the section considered into the area of the section; hence, from Article 380 and the two preceding articles, we have

$$\begin{aligned} Q_1 &= v_1 S_1 \\ &= m_2 v . m_1 S \\ &= m_1 m_2 . vS \\ &= m_1 m_2 Q; \end{aligned}$$

which, compared with the preceding value of Q_1, gives

$$m = m_1 m_2.$$

From this we find

$$m_2 = \frac{m}{m_1};$$

from which the coefficient of the mean velocity may be found from the coefficients of discharge and contraction.

For an orifice in a very thin plate, ... $m = 0.62$
For a short tube, Fig. 182, ... $m = 0.82$

A comparison of these results shows that the effect of the short tube is to reduce the amount of contraction (provided there is one in the tube), but that the interference of the particles or filaments still reduces the velocity of discharge. If a small hole be made in the side of the tube, at a distance from the inside of the vessel equal to the radius of the orifice, air will rush into the tube, showing that there is a negative pressure on the tube. If a pipe be attached to the tube so as to cover the hole and extend down into a vessel of water, the water will rise in the tube to balance the negative pressure, the height, according to Article 385, being nearly equal to

$$\frac{(0.18v)^2}{2g};$$

in which 0.18 equals 1—0.82.

394. Large Orifices.—*To find the mean velocity of discharge through a large orifice in the base of a vessel.*

If the orifice is so large compared with the cross section of the vessel as to cause a perceptible velocity of the upper surface of the liquid, the mean velocity of discharge may exceed that due to the head; for all the particles will have an initial velocity which is itself equivalent to a head cor-

responding to that velocity. Therefore, the head due to the discharge will be the head of the liquid in the vessel, *plus* the head due to the velocity of the surface.

Let s be the section of the orifice,

S_1, that of the vessel at the surface of the liquid,

v and v_1, the corresponding velocities.

Then, according to Article 382, we have

$$v_1 = \frac{s}{S_1} v,$$

and the head due to this velocity will be, according to Article 385,

$$\frac{\left(\frac{s}{S_1} v\right)^2}{2g},$$

which, added to the head h of the liquid, gives, according to Article 386,

$$v = \sqrt{2g\left(h + \frac{\left(\frac{s}{S_1} v\right)^2}{2g}\right)},$$

from which we readily find

$$v = \sqrt{\frac{2gh}{1 - \frac{s^2}{S_1^2}}}.$$

If $s = S_1$, $v = \infty$; that is, the velocity must be infinite in order that the section of the issuing vein at the orifice shall equal that at the surface of the liquid.

If s is so small compared with S_1 that it may be neglected, then we have

$$v = \sqrt{2gh},$$

as found in Article 383.

395. External Pressures Considered.

Let p be the pressure per unit of area on the issuing vein due to the atmosphere or other fluid, and

p_1, the pressure per unit of area on the upper surface of the liquid;

then the head due to the difference of these pressures will be (Article 386),

$$h_1 = \frac{p_1 - p}{gD},$$

which must be added to the head of the liquid; hence

$$v = \left[\frac{2g\left(h + \frac{p_1 - p}{gD}\right)}{1 - \frac{s^2}{S_1^2}} \right]^{\frac{1}{2}}.$$

A discussion of this equation will give several of the preceding ones.

[We cannot follow these modifications further in an elementary work, but will add that the formulas have been founded on the hypothesis that the velocity of the particles at their exit was generated in an infinitesimal of space (Article 383), but it is evident that, in a perfect fluid, all the particles in the vessel will be put in motion as soon as the liquid begins to flow. If the vessel be prismatic, and all the horizontal sections are assumed to remain horizontal, and the vessel kept constantly full, the velocity of the particles in the upper surface of the liquid being zero, and t the time for a particle in the upper surface to reach the orifice; then it is found by means of the Calculus that the velocity of exit will be

$$v_1 = \frac{e^{\frac{S_1^2 - s^2}{s^2 h} \cdot vt} - 1}{e^{\frac{S_1^2 - s^2}{s^2 h} \cdot vt} + 1} v$$

in which v has the value given in the preceding equation, and e is the base of the Naperian logarithms.]

396. A Weir is an opening in the side of a vessel for the discharge of a liquid, in which the upper surface of the liquid is a *free surface.*

397. A Rectangular Notch.—To find the quantity of liquid which will flow *from a rectangular notch in the side of a vessel, the vessel being kept constantly full.*

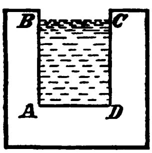


FIG. 183.

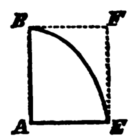


FIG. 184.

Let $ABCD$ be the notch, and x the distance of any filament from the surface BC. The velocity of any fillet will be

$$v = \sqrt{2gx};$$
$$\therefore v^2 = 2gx;$$

which is the equation of a parabola, v being an ordinate, x an abscissa, and $2g$ the parameter. Let $h = AB$, then the velocity of the liquid at A will be

$$v = \sqrt{2gh}.$$

In Fig. 184, take $AE = \sqrt{2gh}$, and construct the parabola BE, then will the velocity at any point in the vertical AB be represented by an ordinate of the parabola drawn through that point. The quantity which will flow through the orifice in a unit of time will be represented by the area of the parabola ABE, multiplied by the width $b = BC$, Fig. 183; hence in a time t it will be represented by the area of the parabola multiplied by the breadth

of the weir and by the time t. But the area of a parabola is two-thirds the area of its circumscribed rectangle; hence we have for the quantity discharged in the time t,

$$Q = m.\tfrac{2}{3}\sqrt{2gh}.h.b.t$$
$$= \tfrac{2}{3}mbt\sqrt{2gh^3}.$$

398. The mean velocity *of the discharge through a rectangular notch at the contracted section* will be

$$v = \frac{Q}{mbht} = \tfrac{2}{3}\sqrt{2gh};$$

that is, two-thirds of the maximum velocity through the notch.

399. The coefficient of discharge over a weir depends upon the head. If the head is very small the coefficient will be small; but for ordinary cases we have

$$m = 0.62 \text{ nearly.}$$

400. Flow through a submerged rectangular orifice *in the side of an upright vessel.*

Let h be the head above the base of the orifice,
h_1, the head above its upper end, and
b, its width.

The discharge will be the same as that due to the difference of two weirs having the same breadth b, and heads h and h_1 respectively; hence the formula of Article 397 becomes

$$Q = \tfrac{2}{3}mbt\sqrt{2g}(h^3 - h_1^3).$$

401. Flow in Long Pipes.—The *law* of resistance in long pipes is rather assumed than deduced. It is assumed in regard to the velocity, that the resistance due to the adhesion of the liquid to the sides of the pipe varies as

the square of the velocity, and that due to viscosity varies directly as the velocity, so that if a and b are two constants, the law will be expressed by

$$av^2+bv.$$

In regard to the dimensions of the pipe the resistance will vary directly as the length and also as the contour (or wetted perimeter) of the pipe, and it is also assumed to vary inversely as the area of the cross section of the stream.

If l be the length of the pipe, S the cross section, and c the contour, or wetted perimeter, then the law will be expressed by

$$\frac{cl}{S}(av^2+bv),$$

or

$$a\frac{cl}{S}(v^2+b'v)$$

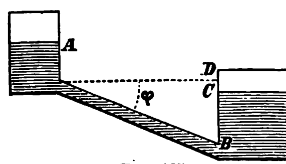


FIG. 185.

in which b' equals $b \div a$.

THE TOTAL HEAD WILL BE the head $AD = h$, of the upper reservoir, *plus* the head due to the fall, or $BD \sin \phi$, *minus* the head in the lower reservoir, if there be one. The total head, *minus* the head due to the velocity of discharge, will equal the total resistance. If the pipe be prismatic and full, the velocity will be uniform throughout its length.

Let $H =$ the total head,

$l = DB$, the length of the pipe,

$\phi =$ the inclination of the pipe to the horizontal,

$h = AD$, and $v =$ the velocity of discharge;

n, if the discharge be into the air, we have

$$H = h + l \sin \phi,$$

and

$$H - \frac{v^2}{2g} = a\frac{cl}{S}(v^2 + b'v).*$$

Numerous experiments have been made by Europ engineers to determine the constants a and b', am which those of Prony, Bossut, and Eytelwein are am the most noted. According to the results of these exp ments, D'Aubisson, a French writer, finally wrote equation as follows:

$$H - \frac{v^2}{2g} = 0.000104392\frac{cl}{S}(v^2 + 0.180449v).$$

402. Circular Pipes.—The section of pipes being cular, if D be the diameter, we have

$$s = \tfrac{1}{4}\pi D^2,$$

and

$$c = \pi D,$$

and the preceding equation becomes

$$H - 0.015536v^2 = 0.000417568\frac{l}{D}(v^2 + 0.180449v).$$

If the quantity of discharge be given, the velocity ma eliminated.

Let Q = the quantity discharged, then

$$Q = \tfrac{1}{4}\pi D^2 v;$$

$$\therefore v = 1.27324\frac{Q}{D^2};$$

which, substituted in the preceding equation, gives

* For a history of this and other formulæ pertaining to the flo water in streams, see *Report on the Hydraulics of the Mississippi R* by Humphreys and Abbott, pp. 207 to 220.

$$H - 0.025187\frac{Q^2}{D^4} = 0.0006769\frac{l}{D^5}(Q^2 + 0.141724QD^2).$$

403. To find the diameter of the pipe *that will give a given discharge.* As D is involved to the fifth power, it is not practicable to make a direct solution. Omitting the terms of least value, that is, the second terms in each member, and we have the following approximate value:

$$D = 0.2323\sqrt[5]{\frac{lQ^2}{H}},$$

which, for velocities above two feet per second, is considered sufficiently accurate.

404. Condition of the Pipe.—The experiments of Mons. Darcy showed that cast-iron pipes, whose interior surface was covered with deposits, offered a much greater resistance than new and clean cast-iron ones, and that when the internal surface was covered with bitumen, or, in other words, was practically polished, the resistance was least.

Bends in pipes also diminish the velocity, and sharp turns offer much greater resistance than rounded ones.

405. Flow in Rivers and Canals.—The formula for the flow in rivers and canals is of the same general form as that given in Article 401 for the flow in long pipes, except that when a portion only of the length of the stream is considered, and the mean velocity is constant, the head due to the terminal velocity will be neglected; for the initial velocity will be the same as the terminal.

Using the constants which were determined by Prony for this case, we have

$$H = 0.000111415\frac{cl}{S}(v^2 + 0.217786v);$$

in which c is the *wetted perimeter*, that is the line in the cross-section which is in contact with the water, and H is the fall for the length l.

Let $Q = vS =$ the quantity of flow, and substituting in the preceding equation

$$v = \frac{Q}{S},$$

and omitting the last term, we find

$$Q = 94.738S\sqrt{\frac{SH}{cl}}.$$

406. Character of the Bed of the Stream.—Mons. Darcy found that streams having cement beds offered the least resistance to the motion, and that the resistance increased in the order of the following substances: Cement, planks, bricks, gravel, and coarse pebbles. (See Morin's *Hydraulique*, Troisième Ed., p. 147.)

407. Cross Section of the Stream.—It is found by observation that the surface of a stream is not horizontal in its cross section, but that it is highest where the velocity

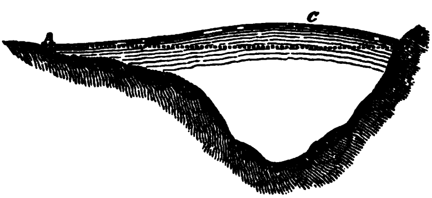


FIG. 186.

is greatest. This is accounted for by the fact that when the fluid is in motion it does not exert as great a side pressure as when it is at rest; and as the velocity near the shore is very small, it requires a greater head near the

middle of the stream, where the velocity is greatest, to balance the pressure at the sides. It may also be observed that, in order to produce a velocity, there will be a greater pressure in the direction of motion in order to overcome the resistances to motion, than there will be in a transverse direction.

408. Backwater caused by a Dam in a Stream.—If a dam be made across a stream, or partly across it, so as to elevate the surface at the place of the dam, the surface of the water above the dam will not be horizontal. If a horizontal line *CK*, Fig. 187, be drawn through the crest of the dam, the surface of the water in the pond will

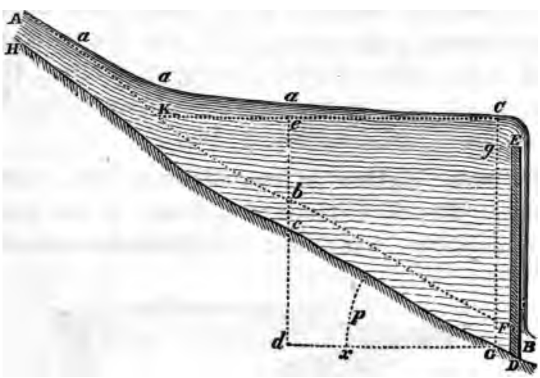


FIG. 187.

be entirely above the line, the difference between the surface and line being very small at first, but increasing gradually as the distance from the dam increases. The natural surface may also be elevated for a long distance back of the point *K*, where the horizontal through *C* intersects the natural surface of the stream. The elevation of the surface above the horizontal *CK*, including also the

elevation back of the point K, is called *backwater*, and is also sometimes called a *remou*.

Backwater is caused partly by the inertia of the liquid and partly by its viscosity. As the stream approaches the dam its velocity is checked, because the pressure on the front side of a particle exceeds that on the rear side, and when the velocity is thus reduced the particles offer a resistance to those which succede, and thus the resistance is, so to speak, extended up the stream. The resistance due to viscosity still further increases this effect.

Fig. 187 shows a section of the river Weser, in Germany, at the place of a certain dam, but the horizontal scale is much less than the vertical. The mean width of the stream was 354 feet, the mean depth about 2.47 feet, the depth of the water just above the dam was 9.82 feet, and hence the surface was raised 7.35 feet. The slope of the stream was quite uniform for a distance of ten miles, and averaged 0.000454 per foot. At the point K, where the horizontal CK, through the crest of the dam, intersected the natural surface of the stream, it was found by actual measurement that the surface was elevated over 15 inches. The distance CK was over three miles. At a distance of four miles above the dam, or about one mile above the point K, the elevation of the surface caused by the dam was about nine inches.

In ordinary streams the width, depth, and the character of the bed are such variable quantities that the extent of the backwater cannot be very accurately computed on a theoretical basis, but empyrical formulas have been given which will give an approximate result when applied to rivers of about the dimensions of the Weser. (See D'Aubisson's *Hydraulics*, Article 166.)

409. Backwater in Rapid Streams.—If the stream is

very rapid, or rapid compared with its depth, the *remous* will be modified, presenting an appearance similar to that shown in Fig. 188. In this case there is a comparatively sudden change in the velocity at the head of the pond.

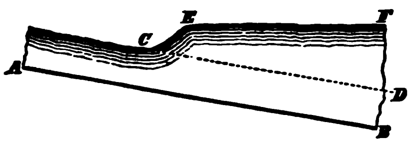


FIG. 188.

Problems.

1. *A prismatic vessel is kept constantly full of a liquid; required the time of discharging a given quantity through a small orifice in the base.*

Let Q be the quantity, h, the height of the liquid, s, the area of the orifice, and t the time; then

$$Q = msvt$$
$$= mst\sqrt{2gh}.$$

From this we find the time of discharge to be

$$t = \frac{Q}{ms\sqrt{2gh}}.$$

2. *Determine the time in which a prismatic vessel will empty itself by an orifice in the base.*

Let S be the area of the free surface of the liquid, s that of the orifice, x the variable head, v the velocity of discharge, and V the velocity of descent of the free surface of the liquid. Then

$$V = \frac{s}{S} v = \frac{s}{S} \sqrt{2gx};$$

$$\therefore V \propto \sqrt{x};$$

which, compared with equation (3), page 34, shows that the law of descent of the surface is the same as that of falling bodies, or rather, it is the same as that of a body projected vertically upward; and from equation (4) on page 34 we have

$$t = \frac{2h}{v},$$

in which h is the height to which a body will rise when projected vertically upward with a velocity v. Were the velocity to remain uniform from the instant that it is projected, the time required to go the same distance would be

$$t_1 = \frac{h}{v};$$

$$\therefore t = 2t_1;$$

hence the time required for the vessel to empty itself will be twice that required to discharge the same quantity when the vessel is kept constantly full. If Q be the contents of the vessel, we have, by multiplying the last equation of the preceding article by two,

$$t = \frac{2Q}{ms\sqrt{2gh}}.$$

3. *To determine the form of a vessel such that the free surface of the liquid shall descend uniformly as it discharges itself through a small orifice at the lower end of the vessel.*

Let the vessel be one of revolution, h the height of the

upper base above the orifice, r the radius of the upper base, x the height of any section above the orifice, y its radius, and $\overline{\Delta x}$ the thickness of the horizontal laminæ; then the conditions of the problem require that the times of descent through the small distance $\overline{\Delta x}$ shall be the same for all positions of the upper surface. Let s be the area of the orifice, and t the time of discharging a quantity equal to any lamina, then

$$ms\sqrt{2gh}.t = \pi r^2.\overline{\Delta x},$$

and

$$ms\sqrt{2gx}.t = \pi y^2.\overline{\Delta x};$$

and dividing one of these equations by the other and reducing, gives

$$x = \frac{h}{r^4}y^4,$$

which is the equation of a biquadratic parabolæ. *Clepsydras* or *Water Clocks* involve these principles. If the time T of the complete discharge of the vessel be given, then will

$$\frac{h}{T} = \frac{\overline{\Delta x}}{t} = \frac{ms\sqrt{2gh}}{\pi r^2} = c,$$

in which c is a constant and is the *rate* of discharge. From this equation we find

$$\frac{h}{r^4} = \frac{\pi^2 c^2}{2gm^2 s^2};$$

and the equation of the curve becomes

$$y = \sqrt[4]{\frac{2gm^2 s^2 x.}{\pi^2 c^2}}$$

Making $x = n$, we have, for the radius of the upper base of the vessel,

$$y = r = \sqrt[4]{\frac{2gm^2s^2h}{\pi^2c^2}}.$$

EXAMPLES.

1. A cylindrical vessel, whose height is 3 feet, radius of the base 6 inches, is filled with water, and discharges itself through an orifice in the base; if the diameter of the orifice is one-half of an inch, and the coefficient of discharge, m, is 0.62, in what time will the vessel empty itself?
2. What quantity of water will flow over a weir whose breadth is 2 feet and constant depth 10 inches, in 45 minutes?
3. It is required to construct a water clock which will empty itself in 10 minutes, the surface descending uniformly. The height of the vessel being 24 inches and radius of its upper base 3 inches; required the equation of the curve, and the area of the orifice, the coefficient of discharge being 0.62.
4. What quantity of water will, in one hour, flow through a pipe 1,500 feet long, 2 inches in diameter, the open end being 25 feet below the level of the reservoir?

EXERCISES.

1. If a vessel having an orifice in its base be filled successively with alcohol, water, and mercury, which will require the least time to empty itself, considering each liquid as *perfect?*
2. If a cylindrical vessel be half filled with mercury and the remaining half with water, will the velocity of discharge of the mercury

through an orifice in the base of the vessel be the same as if the vessel were entirely filled with mercury?

3. If the lower half is water and the upper half mercury, will the flow be the same as in the preceding exercise?

4. If two liquids of different densities are thoroughly mixed, will the velocity of flow from an orifice in a vessel be the same as for each liquid separately?

5. A block is floating on the water in a vessel, when an opening is suddenly made in the base of the vessel; considering that the surface of the water falls with a decreasing acceleration, will the depth of flotation of the block, during the discharge of the water, be the same as before the discharge began? Will the depth of flotation remain constant during the discharge?

6. If, in a pond which receives no supply, an opening is made in one side so that the water will flow out, will the surface remain at a true level—that is, parallel to the surface as it stood before the opening was made?

7. If a vessel filled with water, and having an orifice near its bottom, is placed on a platform and made to ascend with a uniform acceleration, will the velocity of flow through the orifice be the same as if the vessel were at rest?

8. In Fig. 166, if there be an orifice in the base near the outer edge, will the velocity of discharge be the same when the vessel is rotating as when it is at rest?

9. When water is flowing uniformly in a pipe is the pressure against the sides of the pipe the same as if the discharge be stopped?

CHAPTER XXII.

GASES AND VAPORS.

410. A Gas is a fluid whose particles are in a constant state of repulsion. Common air is taken as a type of gases.

411. Pressure of the Atmosphere.—If a tube, 32 or 33 inches long, be closed at one end and filled with mercury, and the open end be closed with the finger until the tube is inverted and the end submerged in a vessel of mercury, then if the finger be removed the mercury in the tube will fall to some point B, and remain nearly stationary. The column AB is sustained by the pressure of the atmosphere upon the surface of the mercury in the vessel A; and hence the weight of a column of mercury equal to AB, having a square inch for its base, will equal the pressure of the atmosphere upon a square inch of surface. The average height of the column AB at the level of the sea is about 29.92 inches (say 30 inches, or 760 millimeters) of mercury, or about 34 feet of water. Hence the mean pressure of the atmosphere, at the level of the sea, is about

Fig. 189.

14.7 pounds (say 15 pounds) per square inch. This is called the pressure of *one atmosphere.* The pressure of the atmosphere diminishes as the distance above the earth increases, and increases for depths below the surface.

412. The Barometer.—If the tube and vessel shown in Fig. 189 be encased in such a way as to retain their relative positions while they are carried from place to place, and the tube be provided with a suitable scale for reading the height of the end B of the mercurial column, above the surface A, the instrument is called a *barometer.* By means of it the pressure of the atmosphere may be readily determined for any place and at any time. There are numerous modifications in the details of different barometers which are explained in descriptive works upon the subject.

413. Height of a Homogeneous Atmosphere.—If the atmosphere were of uniform density, and the same as that at the level of the sea, its height would be found by multiplying 30 inches (the height of the mercurial column) by the ratio of the density of mercury to that of air. Mercury is about $13\frac{1}{2}$ times as dense as water, and water 773 times as dense as air when the barometer stands at 30 inches; hence the height would be

$$2\tfrac{1}{2} \times 773 \times 13\tfrac{1}{2} = 26088 \text{ feet nearly} = 5 \text{ miles nearly.}$$

But the *actual* height is supposed to be from 50 to 100 miles. This is determined by its effect in deflecting the rays of the sun.

414. Boyle's (or Mariotte's) Law.—Boyle in England and Mariotte in France, independently of each other,* demonstrated the following law:

* Writers differ in regard to the respective dates of the discovery. While some state that both made the discovery at about 1668, others give to Boyle a precedence of several years over Mariotte.

For the same temperature the density of a gas is directly proportional to its pressure.

Both these discoverers proved this law by means of a tube, called Mariotte's tube, Fig. 190. Mercury was poured into the tube until the air passage from the short to the long tube was just cut off. This point was marked zero, and the pressure of the air in the short tube was that of one atmosphere when the mercury stood at this point. Mercury was then poured into the long tube until the air in the short column was compressed to one-half its length, when it was found that the upper end of the long column was about 30 inches above the upper end of the mercury in the short tube.

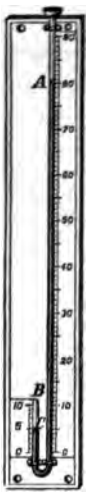


FIG. 190.

Again, filling the long tube until the air in the short tube is compressed to one-quarter of its length, it will be found that the column in the long tube above that in the short tube will be twice its former length, and so on, observing in each case that the temperature must be the same. The quantity of air being constant, the density will be inversely as the volumes, or directly as the pressure.

Let V and V_1 be the volumes corresponding to pressures p and p_1 per square inch, and D and D_1 their densities, then

$$\frac{V}{V_1} = \frac{p_1}{p} = \frac{D_1}{D}$$

$$\therefore V = \frac{V_1 p_1}{p}.$$

If V_1 and p_1 are known, the volume V may be found for

ɔressure p. Considering V and p as variables, the ›ding equation will be of the form

$$xy = m,$$

h is the equation of an hyperbola referred to its ptotes.

5. Boyle's Law is only approximately correct.— nany years after the announcement of Boyle's law it onfirmed by different experimenters, and the law durıat time was supposed to be rigorously correct, but recently the more precise experiments of Despertz Regnault have shown that it differs for different gases is not rigidly true for any gas. But the departure the law is so slight that, for ordinary ɔses, it may safely be considered as .

3. Manometers are instruments for uring the tension of a gas. The *tenis* the pressure per square inch, and is compared with the pressure of one sphere. The principle of the manor is founded on Boyle's law of the comibility of gases. There are many kinds, he closed manometer, Fig. 191, is one ɜ most common. It consists of a vertube closed at the upper end, the lower ɔpening into a vessel of mercury or liquid. Another tube connects the l with the vessel containing the gas, so :he pressure of the gas when it is add into this tube will force the liquid up ›ng one, thereby compressing the air above the liquid. vertical tube is provided with a scale for indicating

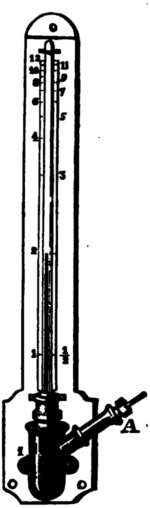


FIG. 191.

the pounds of pressure per square inch, or the number of atmospheres, or both, as may be desired.

417. Expansion of Gases due to a Change of Temperature.—Reliable experiments show that the expansion of all gases under constant pressure may be considered as the same for each degree of increase of temperature for all ranges of temperature. Still the delicate experiments made by Regnault show that the expansions are not identically the same, and that the increase of volume increases somewhat more rapidly than the increase of temperature.

Assuming that Mariotte's law is rigorously exact, and that the rate of expansion of a gas is the same as that of the increase of its temperature, it follows that the *tension of a gas under constant volume varies directly as the change of temperature.*

418. Coefficient of Expansion due to Temperature.—The volume of dry air under a constant pressure increases 0.002039 of its original volume for each increase of 1° Fahrenheit, and this is called the coefficient of expansion. It is also considered as the coefficient of tension under a constant volume for each 1° Fahrenheit.

Let a be the coefficient of expansion (or tension), then

$$\begin{aligned} a &= 0.002039 \text{ for } 1^\circ\ F. \\ &= 0.003670 \text{ for } 1^\circ\ C. \end{aligned}$$

419. To find the Volume of a Gas due to a change of Temperature and Pressure.

Let V_0, p_0, t_0, be the initial volume, tension, and temperature, and V_1, p_1, t_1, the corresponding terminal values. Then will the change of temperature be

$$t_1 - t_0,$$

and hence the volume due to this change will be

$$V_1 = \left(1 + a(t_1 - t_0)\right) V_0;$$

and, according to Mariotte's law, if there be a change of pressure at the same time the volume will be

$$V_1 = \left(1 + a(t_1 - t_0)\right) \frac{p_0}{p_1} V_0;$$

$$\therefore \frac{V_1}{V_0} = \left(1 + a(t_1 - t_0)\right) \frac{p_0}{p_1}.$$

Let V_2, p_2, t_2, be the quantities for another pressure and temperature; then

$$\frac{V_2}{V_0} = \left(1 + a(t_2 - t_0)\right) \frac{p_0}{p_2}.$$

Dividing the former equation by the latter, gives

$$\frac{V_1}{V_2} = \frac{1 + a(t_1 - t_0)}{1 + a(t_2 - t_0)} \times \frac{p_2}{p_1}.$$

420. Perfect Gas.—The more rare a gas is the more perfect it is considered to be. *A perfect gas is defined to be one which is destitute of mass.* A perfect gas does not exist, but this ideal gas serves as a standard of comparison for different gases. It may be defined to be *the limit towards which gases approach as they are expanded indefinitely.* The limit towards which *the coefficient of expansion* approaches is not definitely known, but it is assumed by Rankine to be

$$a = 0.0020275 = \frac{1}{493.2} \text{ for } 1^\circ \text{ F.}$$

$$= 0.00365 = \frac{1}{274} \text{ for } 1^\circ \text{ C.}$$

and these are called the coefficients of expansion for a *perfect* gas.

Substituting the former of these in the formula above, and taking the initial temperature at the melting point of ice, in which case $t_0 = 32°$ F., we have

$$p_1 = (461.2 + t_1) p_0 \frac{V_0}{V_1}.$$

If t_1 were taken at 461.2° F. below zero, this expression would vanish. The *ideal* temperature —461.2° F., or —493.2° F. below melting ice, is called the ABSOLUTE ZERO. This temperature cannot be even approximately reached by any known process; but, according to the formula, it is a temperature at which a perfect gas would lose all expansive power.

The *absolute zero* is used because the formulas for the expansion of gases due to temperature are simplified when the temperature is reckoned from that point.

421. Thermometers are instruments for measuring changes of temperature. They are made on the principle of the uniform rate of expansion of liquids. Mercury is most commonly used, but alcohol is sometimes used; and the latter is necessarily used instead of mercury for temperatures below —40° F., for mercury freezes at about that temperature. Three scales are used, viz.: Fahrenheit's, marked F., the Centigrade, marked C., and Réaumur's, marked R.

Fahrenheit chose for the zero of his scale the height of the mercury, which was produced by a mixture of salt and ice. This he believed to be the absolute zero of cold. The height produced by the boiling point of water he marked 212, and divided the space between these points into equal parts. The freezing point of water was 32 divisions above zero.

The zero of the Centigrade thermometer is at the freez-

ing point of water, and the boiling point of water is marked 100, and the space between is divided into equal parts.

In Reaumur's scale the zero is fixed at the freezing point of water, and the boiling point of water is marked 80, the space between being divided into 80 equal parts.

The melting point of ice is now used instead of the freezing point of water, for the latter is not constant. In all thermometers the divisions below zero are considered *minus*.

422. To Convert one Thermometric Scale into another.—The number of divisions between the melting point of ice and the boiling point of water on the three scales is

$$\begin{array}{ccc} \text{F.} & \text{C.} & \text{R.} \\ 180, & 100, & 80; \end{array}$$

or as

$$1, \qquad \frac{5}{9}, \qquad \frac{4}{9}.$$

But the Fahrenheit scale begins 32° below the others; hence, if F°, C°, and R° represent the degrees on the respective scales for the same temperature, we have

$$C^\circ = \frac{5}{9}(F^\circ - 32^\circ); \qquad R^\circ = \frac{4}{9}(F^\circ - 32^\circ);$$

from which we find

$$F^\circ = \frac{9}{5}C^\circ + 32^\circ, \quad F^\circ = \frac{9}{4}R^\circ + 32^\circ, \quad \text{and } C^\circ = \frac{5}{4}R^\circ.$$

423. Compressed Air.—Air, compressed to several atmospheres, may be used instead of steam for driving engines. If the engine is at a great distance from the compressing machine it is a more desirable power than

steam, for steam will condense and thus lose its power, but compressed air may be conducted for miles, if necessary, without any loss of power except that due to the leakage of the pipes in which it is conducted. It is especially serviceable for driving engines under ground. It is the chief power for driving engines in the construction of large tunnels and in underground mining. It was thus used in the construction of the Mount Cenis tunnel in Europe, and the Hoosac tunnel in this country.

When air is suddenly compressed heat is developed, and sometimes the heat becomes so intense as to make the compressor very hot. If this heat is lost while the air is passing from the compressor to the engine, power is lost. Methods, therefore, have been devised for keeping the air as cool as possible during compression. The most practical way is to inject a spray of water into the compressor while the air is being compressed. If the heat that is in the air when it leaves the compressor could be maintained without extra cost, until the air is used, there would be no loss due to the development of the heat, and as the air thus heated has a greater tension, it would be undesirable to reduce the temperature during compression so far as this cause is concerned.

[The following formula gives the relation between the pressure, density, and temperature of air during rapid changes of motion. (See Rankine's *Applied Mechanics*, p. 582):

$$\frac{\tau_2}{\tau_1}=\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}=\left(\frac{\delta_2}{\delta_1}\right)^{\gamma-1};$$

$$\frac{\delta_2}{\delta_1}=\left(\frac{p_2}{p_1}\right)^{\frac{1}{\gamma}};$$

in which τ_1 is the *absolute* temperature of the air (or gas) when the pressure is p_1 and density δ_1, and τ_2, p_2, δ_2, the corresponding quantities in another condition of the gas. γ is the ratio of spe-

cific heat of constant pressure to the specific heat of constant volume, which for a perfect gas is 1.408, which value is sufficiently exact for dry air. The *quantity* of air being constant, the volumes will be inversely as the densities; hence we have

$$\frac{\tau_2}{\tau_1} = \left(\frac{v_1}{v_2}\right)^{\gamma - 1}.$$

Let the volume v_1, at 14.7 pounds, be 100, and the temperature 60° F., then will the absolute temperature be $\tau_1 = 521.2°$. If the air be suddenly compressed, we will have, according to this formula:

Pressure, lbs. per square inch.	Resulting temperatures. Deg. F.	Resulting Volumes. (Relative.)	Relative Volumes under constant temperature. (Boyle's Law.)
14.7	60	100	100
30.0	180	60	49
45.0	259	45	32
60.0	322	37	25
75.0	374	31	20
90.0	420	28	16
105.0	460	25	14

After passing the compressor, if the heat escapes so as to reduce the temperature to 60°, the volumes will be reduced from those in the third column to those in the fourth, and there will be a corresponding loss of tension in the air. (For a mathematical computation for the work lost in using compressed air, see *Engineering and Mining Journal*, July, 1873.)]

424. Steam or other vapor, separated from liquids, may follow the same laws in regard to expansion, temperature, and density, as air and other gases. If, however, the steam be in contact with the water from which it is generated, the temperature cannot be increased without, at the same time, increasing both the tension and density of the steam.

14*

Steam, in this condition, is, for a given temperature, always at its maximum tension, and also at its maximum density. It may be said to be constantly at its dew point. Steam in which both the density and tension change on account of a change of temperature is called *saturated* steam. But steam in which the tension may change with the temperature without changing its density is called *steam gas*. It follows the laws of permanent gases in this respect. The tension of such steam may greatly exceed that of saturated steam for high temperatures, and when thus heated it is called *Superheated Steam*.

Problems.

1. *To find the weight of a cubic foot of air at any temperature and pressure.*

The weight of a cubic foot of air at 32° F., when the barometer is at 30 inches, is 0.08072 pounds avoirdupois. For a given mass of air the weight of a given volume will be inversely as the temperature and directly as the pressure, or generally, inversely as the total volumes of the air under the different conditions. Hence, the weight, W_1, of a cubic foot will become, according to Article 419,

$$\frac{W_1}{W_0} = \frac{V_0}{V_1} = \frac{1}{1+0.002039(t-32^\circ)} \cdot \frac{p_1}{30}$$

$$\therefore W_1 = \frac{0.08072 p_1}{28.0425 + 0.061170 t},$$

in which p_1 is the reading of the barometer.

2. *Find the weight of a cubic foot of steam at any temperature and pressure.*

The density of steam is five-eighths of the density of air for the same tension and temperature; hence, when the pressure is given in *pounds* per square inch, we have

$$W_1 = \frac{\frac{5}{8} \times 0.08072}{1 + 0.002039(t - 32^\circ)} \cdot \frac{p_1}{14.75}$$

$$= \frac{0.00342 p_1}{13.7877 + 0.030075 t}.$$

3. *A spherical air-bubble rises vertically from a depth* h *to the surface of the liquid; required the equation of the curve described by the extremities of a horizontal diameter.*

Let r = the radius at the surface,
y = the radius at a depth x.

The volumes will be inversely as the pressures. At the surface the pressure per unit is represented by a column of water 34 feet high, and at a depth x by $34 + x$; hence,

$$\tfrac{4}{3}\pi r^3 : \tfrac{4}{3}\pi y^3 :: 34 + x : 34;$$

$$\therefore y^3 = \frac{34 r^3}{34 + x};$$

or

$$x = 34\left(\frac{r^3}{y^3} - 1\right).$$

4. *Required the degree of exhaustion from the receiver of an air-pump.*

At each stroke of the piston a quantity of air is removed from the receiver R and the pipe $a\,c$ equal to the volume in either barrel B or B'.

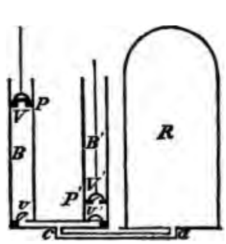


FIG. 192.

Let V be the volume in the receiver and pipes, v, the contents of either barrel, D_0, the initial density, $D_1, D_2 \ldots\ldots D_n$, the densities after the successive strokes.

The quantity remaining after the first stroke will be

$$D_0 V - D_0 v,$$

which will be uniformly distributed throughout the receiver and pipes, and the density will be reduced to D_1; hence

$$D_1 V = D_0 V - D_0 v;$$

$$\therefore D_1 = D_0\left(1 - \frac{v}{V}\right).$$

Similarly, at the end of the second stroke, we have

$$D_2 = D_1\left(1 - \frac{v}{V}\right) = D_0\left(1 - \frac{v}{V}\right)^2;$$

and at the end of the n^{th} stroke

$$D_n = D_0\left(1 - \frac{v}{V}\right)^n.$$

The density D_n cannot be zero so long as n is finite; hence the exhaustion can never be perfect. Indeed, the degree of exhaustion falls far short of that indicated by the formula, for the valves and pistons cannot be made *perfectly* air-tight, and it requires some pressure to operate the valves, so that after a few strokes the exhaustion, in practice, proceeds very slowly.

5. *To determine elevations by means of a barometer.*

In order to solve this problem it is necessary to know the law of diminution of the pressure of the atmosphere. Consider three consecutive strata of the atmosphere of equal thicknesses, but so thin that the density of each may be considered uniform. The pressure on the top of the upper stratum will be the weight of the atmosphere above it, which, if it is near the level of the sea, will be nearly 14.75 pounds. Let this pressure be p_0. The pres-

sure upon the second stratum from the top will be p_1, *plus* the weight of the stratum; hence,

$$p_1 - p_0 = \text{the weight of the first stratum.}$$

Similarly,

$$p_2 - p_1 = \text{the weight of the second stratum.}$$

The weights of the strata are as their densities, or

$$p_1 - p_0 : p_2 - p_1 :: D_0 : D_1.$$

But, according to Mariotte's law, the densities vary directly as the pressures; hence,

$$D_0 : D_1 :: p_0 : p_1;$$

$$\therefore \frac{p_1 - p_0}{p_2 - p_1} = \frac{p_0}{p_1}.$$

These quantities follow the law of a geometrical progression;* hence, the natural numbers, 1, 2, 3, etc., which are as the numbers of the successive laminæ, will be the logarithms of the successive pressures. But the system of logarithms remains to be determined. The thicknesses of the laminæ being constant, the distance below the initial point will equal the thickness of a lamina into the number of laminæ.

---

* Let a be the first term of a geometrical series,
r, the ratio, and
n, the n^{th} term;
then for three consecutive terms we have

$$ar^n,\ ar^{n+1},\ ar^{n+2};$$

then if ar^n be substituted for p_0, ar^{n+1} for p_1, and ar^{n+2} for p_2, the expression in the text will reduce to the identical equation

Let $\overline{\Delta a}$ = the thickness of a lamina,
z = the number of laminæ,
$h = z.\overline{\Delta a}$ = the distance from the initial point,
p_0 = the pressure at the initial point,
p = the pressure at the distance h from the initial point,
a = the base of the system of logarithms, and
m and n constants to be determined;

then the *form* of the expression for the law of pressures will be

$$\log_a \frac{p}{m} = nh;$$

which may be written

$$p = ma^{nh}.$$

Thus far the distance has been reckoned downward. If it be reckoned above the initial point, the sign of h will be changed, and we have

$$p = ma^{-nh}.$$

In this equation, if $h = 0$, the pressure will be represented by p_0; hence,

$$p_0 = m,$$

and the equation becomes

$$p = p_0 a^{-nh}.$$

Let b_0 be the reading of the barometer at the initial station and b the reading at the second station, then

$$\frac{b}{b_0} = \frac{p}{p_0},$$

and the preceding equation becomes

$$b = b_0 a^{-nh};$$

hence, taking the logarithms of both sides, we find

$$h = -\frac{1}{n}\log_a\frac{b}{b_0}$$

$$= \frac{1}{n}\log_a\frac{b_0}{b},$$

in which the subscript, a, indicates that the base of this system is a. To find n and a requires at least two observations at two known elevations; * but, without attempting to find them in this place, we observe that it has been found that the base is that of the Naperian system of logarithms, and

$$\frac{1}{n} = H,$$

---

* These relations are easily deduced in the following manner: Let p be the pressure due to one atmosphere, whose height is H and density D, and dp the pressure due to a lamina whose thickness is dx. Then

$$p = DH,$$

and

$$dp = Ddx.$$

Dividing the latter by the former gives

$$\frac{dp}{p} = \frac{dx}{H};$$

integrating which between the limits of p_0 and p, and 0 and x, gives

$$\log p_0 - \log p = \frac{x}{H};$$

or

$$\log\frac{p}{p_0} = -\frac{x}{H};$$

$$\therefore p = p_0 e^{-\frac{x}{H}} = p_0 e^{-\frac{D_0}{p_0}x};$$

$$\therefore x = H\log\frac{p_0}{p}.$$

the height of a homogeneous atmosphere, which, according to Article 413, is

$$H = 26{,}170 \text{ feet nearly.}$$

If common logarithms are used, they must be divided by the modulus of the common system; or, what is the same thing, multiplied by the reciprocal of the modulus, to reduce the result to an equivalent value. Hence, the value of h becomes

$$h = 2.30258 \times 26170 \log \frac{b_0}{b}$$

$$= 60258 \log \frac{b_0}{b}.$$

By means of actual observations it has been found that the coefficient should be somewhat larger than that given above, and that

$$h = 60346 \log \frac{b_0}{b}$$

gives better results.

It is necessary, however, to add several corrections to this formula. The pressure of the air and the weight of the mercury will both vary with the force of gravity. The force of gravity at any latitude, L, that at 45 degrees being unity, will be (see page 32),

$$g = 1 - \frac{0.08238}{32.1726} \cos 2L$$

$$= 1 - 0.00256 \cos 2L,$$

and the coefficient 60346 must be multiplied by this quantity.

The density of the atmosphere also changes with its temperature. Let t_1 be the temperature at the first sta-

tion and t_2 that at the second, then will the mean temperature be

$$\tfrac{1}{2}(t_1+t_2),$$

which may be considered as the temperature above or below 32°. Hence, the expansion of the air will be represented by the expression

$$1+a\left[\tfrac{1}{2}(t_1+t_2)-32^\circ\right]$$
$$=1+0.002039\left[\tfrac{1}{2}(t_1+t_2)-32^\circ\right];$$

or, for the Centigrade scale,

$$1+0.00366\left[\tfrac{1}{2}(t_1+t_2)\right].$$

Hence, the formula for the height becomes

$$h \text{ (in feet)} = 60346(1-0.00256\cos 2L)\left[1+0.00102(t_1+t_2-64^\circ)\right]\log\frac{b_0}{b_1},$$

which is the formula given by Laplace. But Poisson, in his *Traité de Mécanique*, pages 622–636, introduces corrections for the expansion or contraction of the mercurial column due to changes of temperature, as determined by an attached thermometer. (Mercury expands 0.0001001 of its length for each degree F. of increase of its temperature). He also gives corrections for the diminution of gravity due to an elevation. (It varies inversely as the square of the distance measured from the centre of the earth.) Also a correction due to the attraction of a mountain when observations are taken near it. It is rare, however, that all these refinements are used in practice.

At the level of the sea the mercury stands at		30 inches nearly.	
5,000 feet above	" "	24.7	"
10,000 feet (height of Mt. Ætna)	"	20.5	"
15,000 feet (height of Mt. Blanc)	"	16.9	"
3 miles		16.4	"
6 miles		8.9	"

6. *To find the velocity of discharge of air into a vacuum through a small orifice in a vessel, the pressure within the vessel being equal to one atmosphere.*

This may be solved according to three different hypotheses:

1st. *Suppose that the gas is incompressible.* In this case the head due to the velocity will equal the height of a homogeneous atmosphere, and we have

$$v = \sqrt{2gh} = \sqrt{64\tfrac{1}{3} \times 26170}$$
$$= 1300 \text{ ft. per sec. nearly,}$$
$$= 886 \text{ miles per hour nearly.}$$

2d. *Suppose that the gas is compressible and perfectly elastic.* Then it may be shown by higher analysis that

$$v = \sqrt{2P \log \frac{p}{p_1}},$$

in which P denotes the pressure of one atmosphere, p the pressure within the vessel, and p_1 the external pressure. But when the issue is into a vacuum $p_1 = o$, and we have

$$v = \infty,$$

that is, for this case, according to this hypothesis, the velocity will be infinite, a result which is incorrect, as shown in the following Article.

3d. *Consider the gas not only as elastic and compressible, but also that its temperature and density change suddenly as the gas escapes.* It may be found for this case (see Rankine's *Applied Mechanics*, page 582) that

$$v^2 = \frac{2g\gamma p_0 \tau_1}{(\gamma - 1)\delta_0 \tau_0}\left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right];$$

in which

$\gamma = 1.408$,

$\tau_0 = 493.2°$ F. above absolute zero = the absolute temperature of melting ice,

$\tau_1 =$ the absolute temperature of the gas in the vessel,

$p_0 =$ the height of the mercurial column at the level of the sea,

$\delta_0 =$ the density of air compared with that of mercury,

$p_2 =$ the pressure per unit against the issuing jet, and

$p_1 =$ the pressure per unit within the vessel.

If the jet issue into a vacuum, we have

$$p_2 = 0,$$

and the equation becomes

$$v = \left[\frac{2g\gamma p_0 \tau_1}{(\gamma - 1)\delta_0 \tau_0}\right]^{\frac{1}{2}}.$$

$\frac{p_0}{\delta_0} = 26{,}214$, as given by Rankine, which is the height of a homogeneous atmosphere for *dry* air.

Substituting the several quantities given above in the preceding equation gives

$$v = 2{,}413\sqrt{\frac{\tau_1}{\tau_0}} \text{ feet per second.}$$

At the temperature of freezing, $\tau_1 = \tau_0$, and we have

$$v = 2{,}413 \text{ feet per second.}$$

EXAMPLES.

1. Required the number of degrees through which a given volume of air must be heated so as to double its volume, the pressure remaining constant.

Ans. 490.

2. A tube 30 inches long, closed at one end and open at the other, was sunk in the sea with the open end downward, until the inclosed air occupied only one inch of the tube; what was its depth?

Ans. 986 feet.

3. A spherical air-bubble having risen from a depth of 1,000 feet in water, was one inch in diameter when it reached the surface; what was its diameter at the point where it started? *Ans.* 0.32 inches.

4. A balloon whose capacity is 10,000 cubic feet is filled with hydrogen gas; the specific gravity of the gas being 0.0690 compared with air, how many pounds of ballast will just prevent it from rising?

5. Five cubic feet of air at 32° F. and pressure of 15 lbs. per square inch, is confined in a vessel; what will be the tension when heated to 400° F. and the volume increased to 5.5 cubic feet?

The following examples are selected from the London University Examination Papers, from 1862 to 1870:

1. Explain the difficulty of opening a lock-gate when the water is at a different level within and without the lock; also, why the force required to open the gate is not proportional directly to the difference of level.

2. The weight of water is 770 times that of air; at what depth in a lake would a bubble of air be compressed to the density of water, supposing the law of Mariotte to hold good throughout for the compression?

3. A body weighs in air 1,000 grains, in water 300 grains, and in another liquid 420 grains; what is the specific gravity of the latter liquid?

Ans. .8285.

4. A mercurial barometer is lowered into a vessel of water, so that the surface of the water is finally six inches above the cistern of the

barometer. What *kind* of change will take place in the reading of the column of the instrument? Give a reason for your reply.

5. If a bottle filled with air be tightly corked, and lowered into the ocean, the cork will be forced in at a certain depth. Why is this? and what will take place if the bottle be filled with water instead of air?

6. If a barometer were carried down in a diving-bell, what would take place?

7. A solid, soluble in water but not in alcohol, weighs 346 grains in air, and 210 in alcohol; find the specific gravity of the solid, that of alcohol being 0.85. *Ans.* 2.1625.

8. A body whose specific gravity is 3.5, weighs 4 lbs. in water. What is its real weight? *Ans.* 5.6 lbs.

9. If as much additional air were forced into a closed vessel as it previously contained when in communication with the atmosphere, what would be the pressure on a square inch of the internal surface?

10. At what depth in a lake is the pressure of the water, including the atmospheric pressure, three times as great as at the depth of 10 feet, on a day when the height of the liquid column in a water-barometer is 33 feet 6 inches? *Ans.* 97 feet.

11. A lump of beeswax, weighing 2,895 grains, is stuck on to a crystal of quartz weighing 795 grains, and the whole, when suspended in water, is found to weigh 390 grains; find the specific gravity of beeswax, that of quartz being 2.65. *Ans.* .965.

12. A barometric tube of half an inch internal diameter is filled in the usual way, and the mercury is found to stand at the height of 30 inches. A cubic inch of air having been allowed to pass into the vacuum above the mercury, the column is found to be depressed 5 inches. What was the volume of the original vacuum?

13. A bottle holds 1,500 grains of water, and when filled with alcohol it weighs 1,708 grains; but when empty it weighs 520 grains; what is the specific gravity of alcohol? *Ans.* .792.

14. A piece of cupric sulphate weighs 3 ozs. *in vacuo*, and 1.86 ozs. in oil of turpentine; what is the specific gravity of cupric sulphate, that of turpentine being 0.88? *Ans.* $2\frac{6}{19}$.

15. If the height of the barometer rises from 30 inches to 30.25 inches, what is the *increase* of pressure (in ozs.) upon a square foot?—the

weight of a cubic foot of water being taken to be 1,000 ozs., and the specific gravity of mercury 13.56. *Ans.* 282.5 ozs.

16. A cylindrical vessel standing on a table contains water, and a piece of lead of given size supported by a string is dipped into the water; how will the pressure on the base be affected (1) when the vessel is full, (2) when it is not full? and in the second case, what is the amount of the change?

17. A wooden sphere has a small hole drilled in it, and is placed in water. Find its positions of equilibrium; and state which position is of stable, and which is of unstable equilibrium.

18. The water above the empty lock of a canal is 8 feet higher than the base of the floodgates, which are 4 feet broad, and provided with handles 10 feet long; find what force would have to be applied to the extremity of the handle to force open a floodgate, without previously letting in the water, assuming a cubic foot of water to weigh 1,000 ozs. avoirdupois. *Ans.* 1,600 lbs.

19. A balance is wholly immersed in water, and a body appears to weigh 1 lb., the weights against which it is balanced having the specific gravity 8.5. What will it appear to weigh when balanced against weights of the specific gravity 11.5? *Ans.* $\frac{115}{119}$ lbs.

20. When a body is floating partly immersed in a liquid, what effect will a fall of the barometer have upon the body?

21. The specific gravity of cast copper is 8.79, and that of copper wire is 8.88. What change of volume does a kilogramme of cast copper undergo in being drawn out into wire? *Ans.* 1.15 cubic centimetres.

22. A cylindrical wooden rod of specific gravity .72 and 1 centimetre in diameter is loaded at one end with 9.08 grammes of lead (specific gravity = 11.35); how long must the rod be in order that it may just float in water at the maximum density?

23. State the relation between the pressure and density of an elastic fluid.

24. A piece of cork floats in a basin of water, and the basin is placed under the receiver of an air-pump. State and explain the effect of pumping out a portion of the air in the receiver.

25. A wineglass is inverted, and its rim just immersed in water. What would be the effect of placing a small piece of ice in the water beneath the glass?

26. Find the atmospheric pressure on a square inch, assuming that the height of a column of water supported by the atmospheric pressure is 30 feet, and that a cubic fathom of water weighs six tons.

27. Compare the depths to which a right cone must be immersed in a fluid of twice its density, that it may be in equilibrium when (1) the vertex is downwards, and (2) the base.

28. A floodgate is 6 feet wide and 12 feet deep. Reckoning the weight of a cubic fathom of water at 6 tons, what is the total pressure on the floodgate when the water is level with its top; and what is the situation of the centre of pressure?

29. A cubic inch of one of two liquids weighs a grains, and of the other b grains. A body immersed in the first fluid weighs p grains, and immersed in the second fluid weighs q grains. What is its true weight, and what is its volume?

Ans. $V = (p-q) \div (b-a)$, $W = (bp-aq) \div (b-a)$.

30. A quantity of air contained in a spherical vessel is transferred first into a cylindrical vessel, and then into a cubical vessel, each of which would just circumscribe the spherical vessel. Compare the total pressure produced by the air on the walls of the three vessels.

Ans. $5\frac{1}{3}\pi^2 : 12\pi^2 : 192$.

APPENDIX.

TABLE I.

EXPERIMENTS ON FRICTION, WITHOUT UNGUENTS. BY M. MORIN.

SURFACES OF CONTACT.	Friction of Motion.		Friction of Quiescence.	
	Coefficient of Friction.	Limiting Angle of Resistance.	Coefficient of Friction.	Limiting Angle of Resistance.
Oak upon oak, the direction of the fibres being parallel to the motion..........	0·478	25° 33′	0·625	32° 1′
Oak upon oak, the direction of the fibres of the moving surface being perpendicular to those of the quiescent surface and to the direction of the motion *............................	0·324	17 58	0·540	28 23
Oak upon oak, the fibres of both surfaces being perpendicular to the direction of the motion........................	0·336	18 35	......	......
Oak upon oak, the fibres of the moving surface being perpendicular to the surface of contact, and those of the surface at rest parallel to the direction of the motion......................	0·192	10 52	0·271	15 10
Oak upon oak, the fibres of both surfaces being perpendicular to the surface of contact, or the pieces end to end.....	......	......	0·43	23 17

* The dimensions of the surfaces of contact were in this experiment ·947 square feet, and the results were nearly uniform. When the dimensions were diminished to ·043, a tearing of the fibre became apparent in the case of motion, and there were symptoms of the combustion of the wood; from these circumstances there resulted an irregularity in the friction indicative of excessive pressure.

TABLE I.—*Continued.*

SURFACES OF CONTACT.	FRICTION OF MOTION. Coefficient of Friction.	FRICTION OF MOTION. Limiting Angle of Resistance.	FRICTION OF QUIESCENCE. Coefficient of Friction.	FRICTION OF QUIESCENCE. Limiting Angle of Resistance.
Elm upon oak, the direction of the fibres being parallel to the motion..........	0·432	23° 22′	0·694	34° 46′
Oak upon elm, ditto *	0·246	13 50	0·376	20 37
Elm upon oak, the fibres of the moving surface (the elm) being perpendicular to those of the quiescent surface (the oak) and to the direction of the motion	0·450	24 16	0·570	29 41
Ash upon oak, the fibres of both surfaces being parallel to the direction of the motion..............................	0·400	21 49	0·570	29 41
Fir upon oak, the fibres of both surfaces being parallel to the direction of the motion..............................	0·355	19 33	0·520	27 29
Beach upon oak, ditto................	0·360	19 48	0·53	27 56
Wild pear tree upon oak, ditto.........	0·370	20 19	0·440	23 45
Service tree upon oak, ditto............	0·400	21 49	0·570	29 41
Wrought iron upon oak, ditto †	0·619	31 47	0·619	31 47
Wrought iron upon oak, the surfaces being greased and well wetted..........	0.256	14 22	0·649	33 0
Wrought iron upon elm................	0·252	14 9	.....	......
Wrought iron upon cast iron, the fibres of the iron being parallel to the motion	0·194	10 59	0·194	10 59
Wrought iron upon wrought iron, the fibres of both surfaces being parallel to the motion........................	0·138	7 52	0·137	7 49
Cast iron upon oak, ditto	0·490	26 7	.....	......
Ditto, the surfaces being greased and wetted..............................	......	......	0·646	32 52
Cast iron upon elm.....................	0·195	11 3	......	......
Cast iron upon cast iron...............	0·152	8 39	0·162	9 13
Ditto, water being interposed between the surfaces.........................	0·314	17 26	.....	......
Cast iron upon brass....................	0·147	8 22	.....	......

* It is worthy of remark that the friction of oak upon elm is but five-ninths of that of elm upon oak.

† In the experiments in which one of the surfaces was of metal, small particles of the metal began, after a time, to be apparent upon the wood, giving it a polished metallic appearance; these were at every experiment wiped off; they indicated a wearing of the metal. The friction of motion and that of quiescence, in these experiments, coincided. The results were remarkably uniform.

TABLE I.—*Continued.*

SURFACES OF CONTACT.	Friction of Motion. Coefficient of Friction.	Friction of Motion. Limiting Angle of Resistance.	Friction of Quiescence. Coefficient of Friction.	Friction of Quiescence. Limiting Angle of Resistance.
Oak upon cast iron, the fibres of the wood being perpendicular to the direction of the motion	0·372	20° 25′	......	......
Hornbeam upon cast iron—fibres parallel to motion	0·394	21 31	......	......
Wild pear tree upon cast iron—fibres parallel to the motion	0·436	23 34	......	......
Steel upon cast iron	0·202	11 26	......	......
Steel upon brass	0·152	8 39	......	......
Yellow copper upon cast iron	0·189	10 49	......	......
Do. do. oak	0·617	31 41	0·617	31 41
Brass upon cast iron	0·217	12 15	......	......
Brass upon wrought iron, the fibres of the iron being parallel to the motion	0·161	9 9	......	......
Wrought iron upon brass	0·172	9 46	......	......
Brass upon brass	0·201	11 22	......	......
Black leather (curried) upon oak *	0·265	14 51	0·74	36 31
Ox hide (such as that used for soles and for the stuffing of pistons) upon oak, rough	0·52	27 29	0·605	31 11
Do. do. smooth	0·335	18 31	0·43	23 17
Leather as above, polished and hardened by hammering	0·296	16 30	......	......
Hempen girth, or pulley-band (sangle de chanvre), upon oak, the fibres of the wood and the direction of the cord being *parallel* to the motion	0·52	27 29	0·64	32 38
Hempen matting, woven with small cords, ditto	0·32	17 45	0·50	26 34
Old cordage, 1⅓ inch in diameter, ditto	0·52	27 29	0·79	38 19
Calcareous oolitic stone, used in building, of a moderately hard quality, called stone of Jaumont—upon the same stone	0·64	32 38	0·74	36 31
Hard calcareous stone of Brouck, of a light gray color, susceptible of taking a fine polish (the muschelkalk), moving upon the same stone	0·38	20 49	0·70	35 0

* The friction of motion was very nearly the same whether the surface of contact was inside or the outside of the skin. The *constancy* of the coefficient of the friction of motion was equally apparent in the rough and the smooth skins.

TABLE I.—*Continued.*

SURFACES OF CONTACT.	Friction of Motion. Coefficient of Friction.	Friction of Motion. Limiting Angle of Resistance.	Friction of Quiescence. Coefficient of Friction.	Friction of Quiescence. Limiting Angle of Resistance.
The soft stone mentioned above, upon the hard	0·65	33° 2′	0·75	36° 53′
The hard stone mentioned above upon the soft	0·67	33 50	0·75	36 53
Common brick upon the stone of Jaumont	0·65	33 2	0·65	33 2
Oak upon ditto, the fibres of the wood being perpendicular to the surface of the stone	0·38	20 49	0·63	32 13
Wrought iron upon ditto	0·69	34 37	0·49	26 7′
Common brick upon the stone of Brouck	0·60	30 58	0·67	33 50
Oak as before (endwise) upon ditto	0·38	20 49	0·64	32 36
Iron, ditto ditto	0·24	13 30	0·42	22 47

TABLE II.

EXPERIMENTS ON THE FRICTION OF UNCTUOUS SURFACES. BY M. MORIN.

In these experiments the surfaces, after having been smeared with an unguent, were wiped, so that no interposing layer of the unguent prevented their intimate contact.

SURFACES OF CONTACT.	Friction of Motion.		Friction of Quiescence.	
	Coefficient of Friction.	Limiting Angle of Resistance.	Coefficient of Friction.	Limiting Angle of Resistance.
Oak upon oak, the fibres being parallel to the motion	0·108	6° 10′	0·390	21° 19′
Ditto, the fibres of the moving body being perpendicular to the motion	0·143	8 9	0·314	17 26
Oak upon elm, fibres parallel	0·136	7 45	.....	......
Elm upon oak, ditto	0·119	6 48	0·420	22 47
Beech upon oak, ditto	0·330	18 16	.....	......
Elm upon elm, ditto	0·140	7 59	.....	......
Wrought iron upon elm, ditto	0·138	7 52	.....	......
Ditto upon wrought iron, ditto	0·177	10 3	.....	......
Ditto upon cast iron, ditto	.....	......	0·118	6 44
Cast iron upon wrought iron, ditto	0·143	8 9	.....	......
Wrought iron upon brass	0·160	9 6	.....	......
Brass upon wrought iron	0·166	9 26	.....	......
Cast iron upon oak, ditto	0·107	6 7	0·100	5 43
Ditto upon elm, ditto, the unguent being tallow	0·125	7 8	.....	......
Ditto, ditto, the unguent being hog's lard and black lead	0·137	7 49	.....	......
Elm upon cast iron, fibres parallel	0·135	7 42	0·098	5 36
Cast iron upon cast iron	0·144	8 12	.....	......
Ditto upon brass	0·132	7 32	.....	......
Brass upon cast iron	0·107	6 7	.....	......
Ditto upon brass	0·134	7 38	0·164	9 19
Copper upon oak	0·100	5 43	.....	......
Yellow copper upon cast iron	0·115	6 34	.....	......
Leather (ox hide) well tanned upon cast iron, wetted	0·229	12 54	0·267	14 57
Ditto upon brass, wetted	0·244	13 43	.....	......

TABLE III.

EXPERIMENTS ON FRICTION WITH UNGUENTS INTERPOSED. BY M. MORIN.

The extent of the surfaces in these experiments bore such a relation to the pressure as to cause them to be separated from one another throughout by an interposed stratum of the unguent.

SURFACES OF CONTACT.	FRICTION OF MOTION. Coefficient of Friction.	FRICTION OF QUIESCENCE. Coefficient of Friction.	UNGUENTS.
Oak upon oak, fibres parallel......	0·164	0·440	Dry soap.
Ditto ditto	0·075	0·164	Tallow.
Ditto ditto	0·067	.....	Hog's lard.
Ditto, fibres perpendicular......	0·083	0·254	Tallow.
Ditto ditto	0·072	.....	Hog's lard.
Ditto ditto	0·250	.....	Water.
Ditto upon elm, fibres parallel ..	0·136	.....	Dry soap.
Ditto ditto	0·073	0·178	Tallow.
Ditto ditto	0·066	.....	Hog's lard.
Ditto upon cast iron, ditto......	0·080	.....	Tallow.
Ditto upon wrought iron, ditto..	0·098	.....	Tallow.
Beech upon oak, ditto............	0·055	.....	Tallow.
Elm upon oak, ditto..............	0·137	0·411	Dry soap.
Ditto ditto	0·070	0·142	Tallow.
Ditto ditto	0·060	.....	Hog's lard.
Ditto upon elm, ditto...........	0·139	0·217	Dry soap.
Ditto upon cast iron, ditto......	0·066	.....	Tallow.
Wrought iron upon oak, ditto.....	0·256	0·649	Greased and saturated with water
Ditto ditto ditto	0·214	.....	Dry soap.
Ditto ditto ditto	0·085	0·108	Tallow.
Ditto upon elm, ditto	0·078	.....	Tallow.
Ditto ditto ditto	0·076	.....	Hog's lard.
Ditto ditto ditto	0·055	.....	Olive oil.
Ditto upon cast iron, ditto......	0·103	.....	Tallow.
Ditto ditto ditto	0·076	.....	Hog's lard.
Ditto ditto ditto	0·066	0·100	Olive oil.
Ditto upon wrought iron, ditto..	0·082	.....	Tallow.
Ditto ditto ditto	0·081	.....	Hog's lard.
Ditto ditto ditto	0·070	0·115	Olive oil.
Wrought iron upon brass, fibres parallel.....................	0·103	.....	Tallow.
Ditto ditto ditto	0·075	.....	Hog's lard.
Ditto ditto ditto	0·078	.....	Olive oil.
Cast iron upon oak, ditto.........	0·189	.....	Dry soap.

TABLE III.—*Continued.*

SURFACES OF CONTACT.	FRICTION OF MOTION. Coefficient of Friction.	FRICTION OF QUIESCENCE. Coefficient of Friction.	UNGUENTS.
Cast iron upon oak, fibres parallel.	0·218	0·646	Greased, and saturated with water.
Ditto ditto ditto	0·078	0·100	Tallow.
Ditto ditto ditto	0·075	.....	Hog's lard.
Ditto ditto ditto	0·075	0·100	Olive oil.
Ditto upon elm, ditto	0·077	.....	Tallow.
Ditto ditto ditto	0·061	.....	Olive oil.
Ditto ditto ditto	0·091	.....	Hog's lard & plumbago.
Ditto upon wrought iron.........	.....	0·100	Tallow.
Cast iron upon cast iron...........	0·314	.....	Water.
Ditto ditto	0·197	.....	Soap.
Ditto ditto	0·100	0·100	Tallow.
Ditto ditto	0·070	0·100	Hog's lard.
Ditto ditto	0·064	.....	Olive oil.
Ditto ditto	0·055	.....	Lard and plumbago.
Ditto upon brass...............	0·103	.....	Tallow.
Ditto ditto	0·075	.....	Hog's lard.
Ditto ditto	0·078	.....	Olive oil.
Copper upon oak, fibres parallel...	0·069	0·100	Tallow.
Yellow copper upon cast iron......	0·072	0·103	Tallow.
Ditto ditto	0·068	.....	Hog's lard.
Ditto ditto	0·066	.....	Olive oil.
Brass upon cast iron..............	0·086	0·106	Tallow.
Ditto ditto	0·077	.....	Olive oil.
Ditto upon wrought iron.........	0·081	.....	Tallow.
Ditto ditto	0·089	.....	Lard and plumbago.
Ditto ditto	0·072	.....	Olive oil.
Ditto upon brass................	0·058	.....	Olive oil.
Steel upon cast iron................	0·105	0·108	Tallow.
Ditto ditto	0·081	.....	Hog's lard.
Ditto ditto	0·079	.....	Olive oil.
Ditto upon wrought iron.........	0·093	.....	Tallow.
Ditto ditto	0·076	.....	Hog's lard.
Ditto upon brass................	0·056	.....	Tallow.
Ditto ditto	0·053	.....	Olive oil.
Ditto ditto	0·067	.....	Lard and plumbago.
Tanned ox hide upon cast iron....	0·365	.....	Greased, and saturated with water.
Ditto ditto	0·159	.....	Tallow.
Ditto ditto	0·133	0.122	Olive oil.
Ditto upon brass................	0·241	.....	Tallow.

TABLE III.—*Continued.*

SURFACES OF CONTACT.	FRICTION OF MOTION. Coefficient of Friction.	FRICTION OF QUIESCENCE. Coefficient of Friction.	UNGUENTS.
Tanned ox hide upon brass	0·191	.....	Olive oil.
Ditto upon oak	0·29	0·79	Water.
Hempen fibres not twisted, moving upon oak, the fibres of the hemp being placed in a direction perpendicular to the direction of the motion, and those of the oak parallel to it	0·332	0·869	Greased, and saturated with water.
The same as above, moving upon cast iron	0·194	.....	Tallow.
Ditto	0·153	.....	Olive oil.
Soft calcareous stone of Jaumont upon the same, with a layer of mortar, of sand, and lime interposed, after from 10 to 15 minutes' contact	.....	0·74	

TABLE IV.

OF THE SPECIFIC GRAVITIES OF BODIES.

[The density of distilled water is reckoned in this Table at its maximum, 38¾° F. = 1.000.]

Name of the Body.	Specific Gravity.
I. SOLID BODIES.	
(1) METALS.	
Antimony (of the laboratory)	4·2
Brass	7·6
Bronze for cannon, according to Lieut. Matzka	8·414
Ditto, mean	8·758
Copper, melted	7·788
Ditto, hammered	8·878
Ditto, wire-drawn	8·78
Gold, melted	19·238
Ditto, hammered	19·361
Iron, wrought	7·207

TABLE IV.—*Continued.*

Name of the Body.	Specific Gravity.	
I. SOLID BODIES.		
ast, a mean	7·251	
gray	7·2	
white	7·5	
or cannon, a mean	7·21	
pure melted	11·3303	
flattened	11·388	
m, native	16·0	
melted	20·855	
hammered and wire-drawn	21·25	
ilver, at 32° Fahr	13·568	
pure melted	10·474	
hammered	10·51	
ast	7·919	
wrought	7·840	
much hardened	7·818	
slightly	7·833	
hemically pure	7·291	
hammered	7·299	
Bohemian and Saxon	7·312	
English	7·291	
nelted	6·861	
rolled	7·191	
(2) BUILDING STONES.		
ter	2·7	
	2·8	
e	2·72	
	2·5	
e	2·5	
lende	2·9	
one, various kinds	2·64	
ite	2·51	
ry	2·4	
	2·56	
one, various kinds, a mean	2·2	
for building	1·66	
e	2·5	
te	2·4	
	1·41	
(3) WOODS.	Fresh felled.	Dry.
	0·8571	0·5001
	0·9036	0·6440
	0·7654	0·4302
	0·9012	0·6274
	0·9822	0·5907
	0·9476	0·5474
	0·8941	0·5550
eam	0·9452	0·7695
chestnut	0·8614	0·5749

TABLE IV.—*Continued.*

Name of the Body.	Specific Gravity.	
I. SOLID BODIES.	Fresh felled.	Dry.
Larch	0·9206	0·4735
Lime	0·8170	0·4390
Maple	0·9036	0·6592
Oak	1·0494	0·6777
Ditto, another specimen	1·0754	0·7075
Pine, *Pinus Abies Picea*	0·8699	0·4716
Ditto, *Pinus Sylvestris*	0·9121	0·5502
Poplar (Italian)	0·7634	0·3931
Willow	0·7155	0·5289
Ditto, white	0·9859	0·4873
(4). VARIOUS SOLID BODIES.		
Charcoal, of cork	0·1	
Ditto, soft wood	0·28	
Ditto, oak	1·573	
Coal	1·232	
Coke	1·865	
Earth, common	1·48	
rough sand	1·92	
rough earth, with gravel	2·02	
moist sand	2·05	
gravelly soil	2·07	
clay	2·15	
clay or loam, with gravel	2·48	
Flint, dark	2·542	
Ditto, white	2·741	
Gunpowder, loosely filled in,		
coarse powder	0·886	
musket ditto	0·992	
Ditto, slightly shaken down,		
musket powder	1·069	
Ditto, solid	2·248	
Ice	0·916	
Lime, unslacked	1·842	
Resin, common	1·089	
Rock salt	2·257	
Saltpetre, melted	2·745	
Ditto, crystallized	1·900	
Slate-pencil	1·8	
Sulphur	1·92	
Tallow	0·942	
Turpentine	0·991	
Wax, white	0·969	
Ditto, yellow	0·965	
Ditto, shoemaker's	0·897	
II. LIQUIDS.		
Acid, acetic	1·063	
Ditto, muriatic	1·211	
Ditto, nitric, concentrated	1·521	

TABLE IV.—*Continued.*

Name of the Body.	Specific Gravity.
, sulphuric, English	1·845
,, concentrated (Nordh.)	1·860
ıol, free from water	0·792
,, common	0·824
ıoniac, liquid	0·875
ıfortis, double	1·300
,, single	1·200
	1·023
r, acetic	0·866
,, muriatic	0·845
,, nitric	0·886
,, sulphuric	0·715
inseed	0·928
,, olive	0·915
,, turpentine	0·792
,, whale	0·923
ksilver	13·568
r, distilled	1·000
,, rain	1·0013
,, sea	1·0265
	0·992
III. GASES.	**Barometer 30 In. Temperature = 32°.**
ɔspheric air = $\frac{1}{773}$ =	1·0000
ɔnic acid gas	1·5240
onic oxide gas	0·9569
uretted hydrogen, a maximum	0·9784
from coals	0·3000 / 0·5596
·ine	2·4700
·iodic gas	4·4430
·ogen	0·0688
·osulphuric acid gas	1·1912
atic acid gas	1·2474
ɔgen	0·9760
;en	1·1026
phuretted hydrogen gas	0·8700
n at 212° Fahr	0·6235
hurous acid gas	2·2470

ANSWERS.

Pages 13, 14. **1.** 250 ft. and 50 ft. **2.** 25 ft. and $12\frac{1}{2}$ ft. **3.** 112 ft. **4.** $6\frac{1}{4}$ ft. **5.** $\frac{5}{2}\sqrt{2}$. **6.** 0·6468+. **7.** 9·8.

Pages 35, 36. **1.** 80·2 ft. **2.** 9·3 sec. **3.** 156·4 ft. **4.** 386 ft. **5.** 39·23 m. **6.** 116·33 ft. **7.** 3·15 sec. **10.** $\left(\frac{a}{v+V}\right)\left(V-\frac{1}{2}g\frac{a}{v+V}\right)$

Page 47. **1.** 35·87 ft. **2.** $80\frac{5}{12}$ ft. **3.** 13·88+ft. **4.** $31\frac{1}{4}$ lbs.

Page 50. **1.** $201\frac{1}{24}$ ft. **2.** $134\frac{1}{36}$ ft. **3.** 52·55 lbs. and 47·57 lbs. **4.** 64·6 ft. **5.** $1\frac{5}{9}$ lbs.; $3\frac{1}{3}$ lbs.; 10 lbs.; ∞. **6.** 2·61 sec.

Pages 75, 76. **1.** 3886 ft.–lbs. **2.** 18 ft. **3.** 2·53 miles. **4.** 19·3 lbs. **5.** $3\frac{4}{7}$ ft. **6.** 160·8 ft.

Pages 91, 92. **1.** 25 pounds—feet—seconds. **2.** 0·05538 inches. **3.** 24,446,255 lbs.

Page 99. **7.** Impossible, unless the angle 35° be considered as the supplement of the real angle. If so considered, then angle $(PF) = 30°$. $F = 90·63$ lbs. and $R = 57·34$ lbs. **8.** 12 lbs.; 16 lbs.

Page 103. **2.** 22·36.

Pages 115, 116. **1.** 144·2 lbs. **2.** 120 lbs. **3.** 60 lbs. **4.** $\sqrt{2}AC$. **5.** 86·1 lbs.; 81·16 lbs. **6.** $\frac{BD}{AB\sin BAC}W$. **7.** Bisect the angle. **8.** 931·8 lbs. **9.** ∞.

Page 133. **3.** 14 lbs.; $3\frac{4}{7}$ feet. **4.** 18 lbs.; 4 feet $3\frac{1}{3}$ inches. **5.** $\frac{PF}{P+F}AB\sin\phi$.

Page 154. **2.** $\dfrac{\frac{2}{3}r\dfrac{\sin\phi}{\phi}\times \text{area } AGBC-\frac{2}{3}CH.ABC}{\text{area } AHBG}$. **3.** $\frac{4}{3}\pi\overline{CG}^3$. **4.** $\frac{1}{6}\pi(AB)^3$. **5.** $\frac{2}{3}\pi(AB)(CH)^2$.

Page 169. **1.** 48·6 ft. **2.** (Make $v = 0$ in Eq. 1, p. 165) $P \sin B = W \sin A$. **3.** $P = 1{\cdot}38\, W$. **4.** 8° 37′ 40″. **6.** $ECg = \sin^{-1} \dfrac{4r}{3(R-r)}$.
7. $t = \dfrac{q}{k\sqrt{2gh}}$.

Pages 179, 180. **1.** 26·8 lbs. **2.** 60 lbs. **3.** $W \sin A$. **4.** $W \tan A$.
5. 16·1 lbs.; 34·4 lbs. **6.** 60 lbs. **7.** 9° 35′ 40″. **8.** 28° 57′ 17″.
9. $P = W$; 60°. **10.** $\sqrt{g}$ feet.

Pages 192, 193. **1.** 90·4 lbs. **3.** $W \cot \beta$; $W \operatorname{cosec} \beta$. **4.** $t = 250$ lbs., $c = 354$ lbs. **5.** 30°. **6.** 26° 33′ 54″.

Page 201. **2.** $b = 8{\cdot}10$ in.; $d = 12{\cdot}48$ in. **3.** 9·29 in. **4.** 11111 lbs.
5. 24·5 in. **6.** 76,800 lbs. **7.** 23 ft. 6·8 in.

Pages 208, 209. **1.** 14° 24′. **2.** 47·65 ft.; −34·3 ft.; −637·3 ft.
3. 25·36 ft. **4.** Draw a line from the point to the extremity of the vertical diameter of the circle, and the path required will be the external part of the secant. **5.** 26·05 miles per hour. **6.** 4857·5 ft.
7. 164·5 sec.; 302·8 sec. **8.** 32·34 ft. **9.** 1903·64 ft.

Pages 217, 218. **1.** 804⅙ ft.; 201 ft. **2.** 86·8 ft. **5.** 45°. **6.** 407·87 ft.
7. 17·6 ft. **8.** 228·2 ft. per sec. (velocity of projection); 15° 49′ 9″; 850 ft.; 3·9 sec.

Pages 233, 234. **1.** $v = \sqrt{2gr}$. **2.** 38·3. **3.** 38·3. **4.** $2W$.
5. $\dfrac{30}{\pi}\sqrt{\dfrac{g}{\mu R}}$. **6.** 407·87 ft. **7.** 61·14 m. per. h. **8.** 1·12 in.
9. 1·93 in. **10.** 76° 25′ 40″; $AC = 14{\cdot}58$ in.; $BC = 3{\cdot}52$ inches.
11. 118·95. **12.** Let $\delta =$ the weight of a unit of volume, then the number of revolutions per minute will be $\dfrac{13786 \cdot 7}{\sqrt{\delta}}$. If the material weighs 144 lbs. per cubic foot (which is somewhat two small for most stone, but sufficiently exact for a near approximation), we have 1148·9 turns per minute, or 19·15 turns per second.

Pages 248, 249. **3.** 0.0181 in. **4.** 11024·3 ft. **5.** 3·9168 sec. **6.** 4·9 miles per second. **7.** 42 m. 14 sec. **8.** 0·453 ft. per sec. **9.** 0·0148 second.

Pages 258, 259. **1.** Pressure on the bottom and sides, 44003·16 lbs.
2. 1·736 lbs. **3.** 195·96 lbs. **4.** 28 lbs. **5.** 50·73 ft. per sec.

6. 396.56 lbs. **7.** Let a be the half length of the bar, s the specific gravity of the wood in reference to the liquid, and x the distance from the point of attachment of the cord to the middle point of the bar; then $x = \frac{7s}{18s-4}a$. **8.** 81 inches.

Page 270. **2.** $4\frac{4}{7}$. **3.** 240 grs. **4.** 17968·75 lbs. **5.** $\frac{3}{4}$. **6.** 1·6.

Page 276. **1.** 16° 41′ 57″. **2.** The box being so deep that none of the liquid will flow over, we have $21\frac{1}{3}$ ft. per sec. **3.** g. **4.** 54·16.

Page 282. **2.** 82·28 lbs.; 164·57 lbs. **3.** 1767·1 lbs.; 589·0 lbs.; 589·0 lbs. **4.** 98,175 lbs.; 32,725 lbs. **5.** $4312\frac{1}{2}$ lbs. **6.** 2250 lbs.

Pages 287, 288. **2.** 5·19 in. **3.** 2 ft. $8\frac{2}{3}$ in. **4.** 5.69 ft. **5.** 50,000 lbs.

Page 310. **1.** 6 m. 41·2 sec. **2.** 13618·8 cu. ft. **3.** $x=\frac{8}{9}y^4$; 0·0134 sq. inches. **4.** 39·136 cu. ft.

Page 332. **4.** 751·34 lbs. **5.** 23·868.